

Functional responses, functional covariates and the concurrent model

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1. Predicting precipitation profiles from temperature curves

- Precipitation is much harder to predict than temperature.
- It comes in two main forms:
 - *Drizzle*: Large low pressure zones drop moisture over many hours or days.
 - *Storms*: Convective, short violent storms with a lot of precipitation in a hurry, and spatially localized.
- Precipitation tends to be seasonal; more in the spring and fall than in the summer and winter.

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A model

- We can assume that climate zone is important.
- We will predict log precipitation; logging stabilizes variance and eliminates the positivity constraint.
- We will use the difference $\text{TempRes}_{mg}(t)$ between a temperature profile and the mean for the climate zone as a function covariate.

- We can extend the functional ANOVA model to

$$\log[\text{Prec}_{mg}(t)] = \mu(t) + \alpha_g(t) + \text{TempRes}_{mg}(t)\beta(t) + \epsilon_{mg}(t)$$

- We call this model *concurrent* because it assumes that the temperature today affects today's precipitation.

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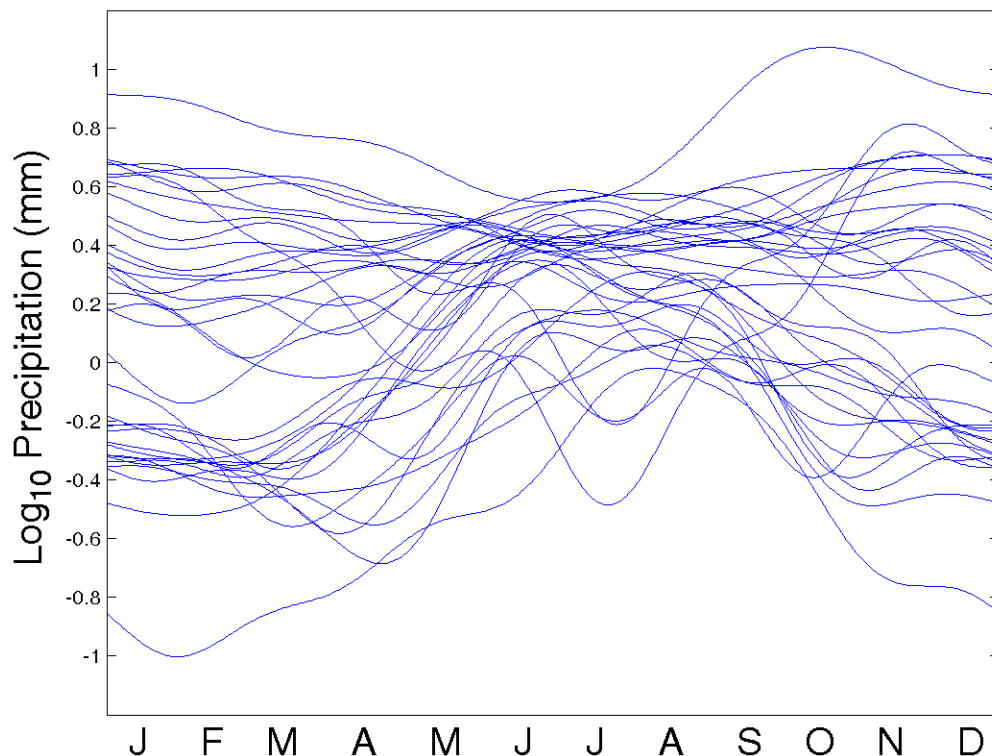
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The functional data

- Where precipitation was recorded as 0 mm, we changed it to 0.05 mm, half the minimum positive value.
- We used 11 Fourier series basis functions for precipitation with no roughness penalty.
- We used 21 Fourier series basis functions for temperature with no roughness penalty.

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Log precipitation profiles



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The fitting criterion and some results

- The fitting criterion was the unpenalized error sum of squares

$$\text{LMSSE}(\mu, \alpha_g, \beta) = \int \sum_{k,g}^N [\text{LogPrec}_{kg}(t) - \mu(t) - \alpha_g(t) - \text{TempRes}_{kg}(t)\beta(t)]^2 dt$$

- The resulting root-mean-squared-residual was 0.19 mm.
- When we dropped $\text{TempRes}(t)$ from the model, this increased to 0.20 mm.
- As we see in the following plot, the only place where temperature appears to make a contribution is in mid-winter.

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The estimated regression function $\beta(t)$

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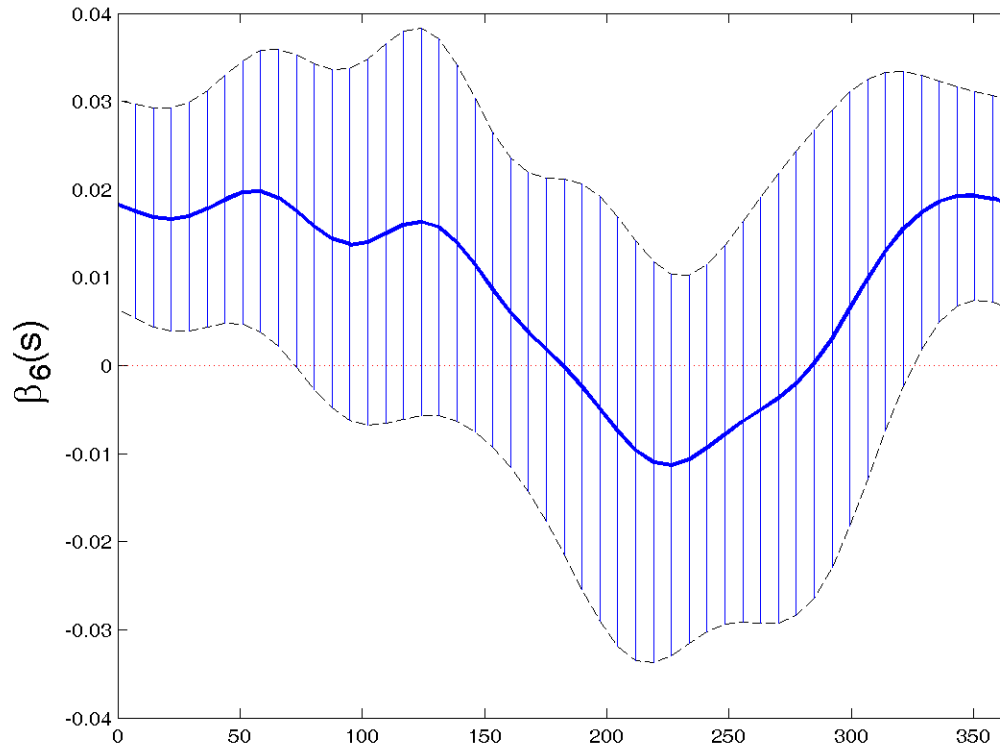
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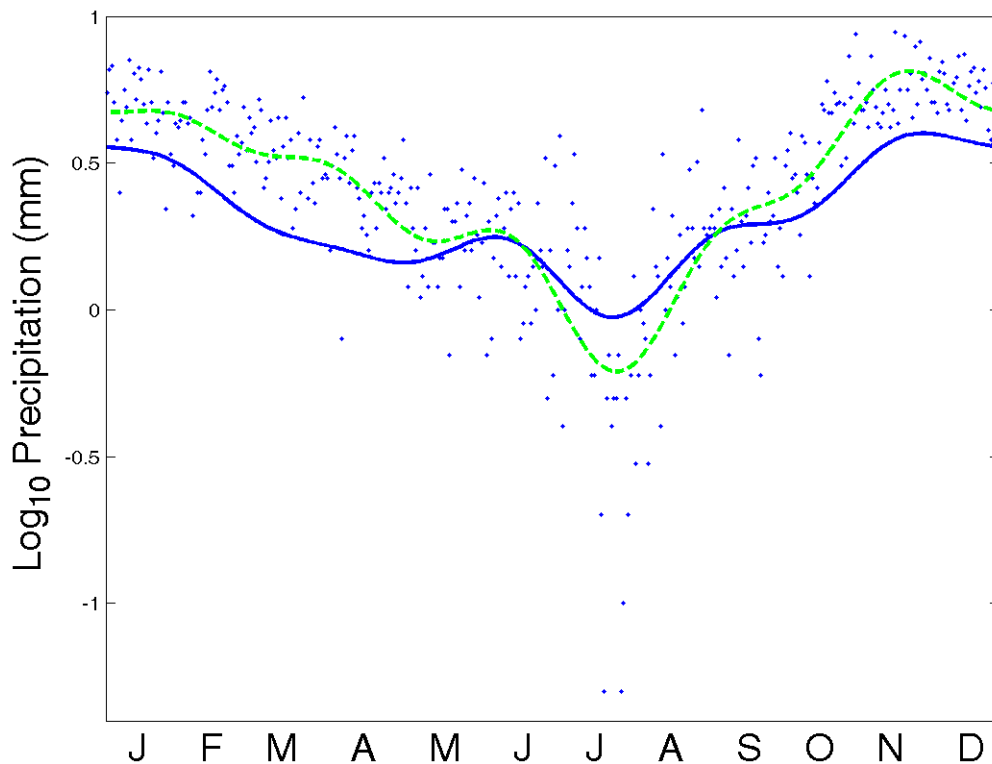
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The fit to Vancouver's data



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A probe for the winter effect

- The confidence limits are point-wise; we need a measure of the temperature influence accumulated over the winter months.
- Here is a probe that works:

$$\int_0^{365} \cos[2\pi(t - 64.5)/365]\beta(t) dt = 2.32 ,$$

- The estimated standard error of this probe is 0.77, giving a t-ratio of 3.0.
- It appears that elevated temperatures in mid-winter go along with increased precipitation.

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2. Fitting the concurrent model

- Here is a general statement of the current functional/functional model:

$$y_i(t) = \sum_{j=1}^q z_{ij}(t)\beta_j(t) + \epsilon_i(t) .$$

- or in matrix notation:

$$\mathbf{y}(t) = \mathbf{Z}(t)\boldsymbol{\beta}(t) + \boldsymbol{\epsilon}(t) ,$$

- We will use a penalized error sum of squares criterion:

$$\begin{aligned} \text{LMSSE}(\boldsymbol{\beta}) = & \int [\mathbf{y}(t) - \mathbf{Z}(t)\boldsymbol{\beta}(t)]' [\mathbf{y}(t) - \mathbf{Z}(t)\boldsymbol{\beta}(t)] dt \\ & + \sum_j^p \lambda_j \int [L_j \beta_j(t)]^2 dt . \end{aligned}$$

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The basis function expansions for $\beta_j(s)$

- Let regression function $\beta_j(s)$ have the expansion

$$\beta_j(s) = \mathbf{b}_j' \boldsymbol{\theta}_j(s)$$

in terms of K_j basis functions $\theta_{jk}(s)$.

- Some of the independent variables can be scalar; in this case the basis for their $\beta_j(s)$'s is the constant basis;

$$\theta_{j1}(s) = 1$$

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- Defining $K_\beta = \sum_j^q K_j$, we construct vector \mathbf{b} of length K_β by stacking the coefficient vectors vertically, that is,

$$\mathbf{b} = (b'_1, b'_2, \dots, b'_q)' .$$

- Now assemble q by K_β matrix function Θ as follows:

$$\Theta = \begin{bmatrix} \boldsymbol{\theta}'_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{\theta}'_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \boldsymbol{\theta}'_q \end{bmatrix} .$$

- We can now express our model as

$$\mathbf{y}(t) = \mathbf{Z}(t)\Theta(t)\mathbf{b} + \boldsymbol{\epsilon}(t) .$$

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- We also need to arrange the order K_j roughness penalty matrices

$$\lambda_j \mathbf{R}_j = \lambda_j \int L\boldsymbol{\theta}_j(t) L\boldsymbol{\theta}'_j(t) dt$$

into the symmetric block diagonal matrix \mathbf{R} of order K_β :

$$\mathbf{R} = \begin{bmatrix} \lambda_1 \mathbf{R}_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 \mathbf{R}_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_q \mathbf{R}_q \end{bmatrix}. \quad (1)$$

The normal equations

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- $$\left[\int \Theta'(t) \mathbf{Z}'(t) \mathbf{Z}(t) \Theta(t) dt + \mathbf{R} \right] \mathbf{b} = \left[\int \Theta'(t) \mathbf{Z}'(t) \mathbf{y}(t) dt \right]$$

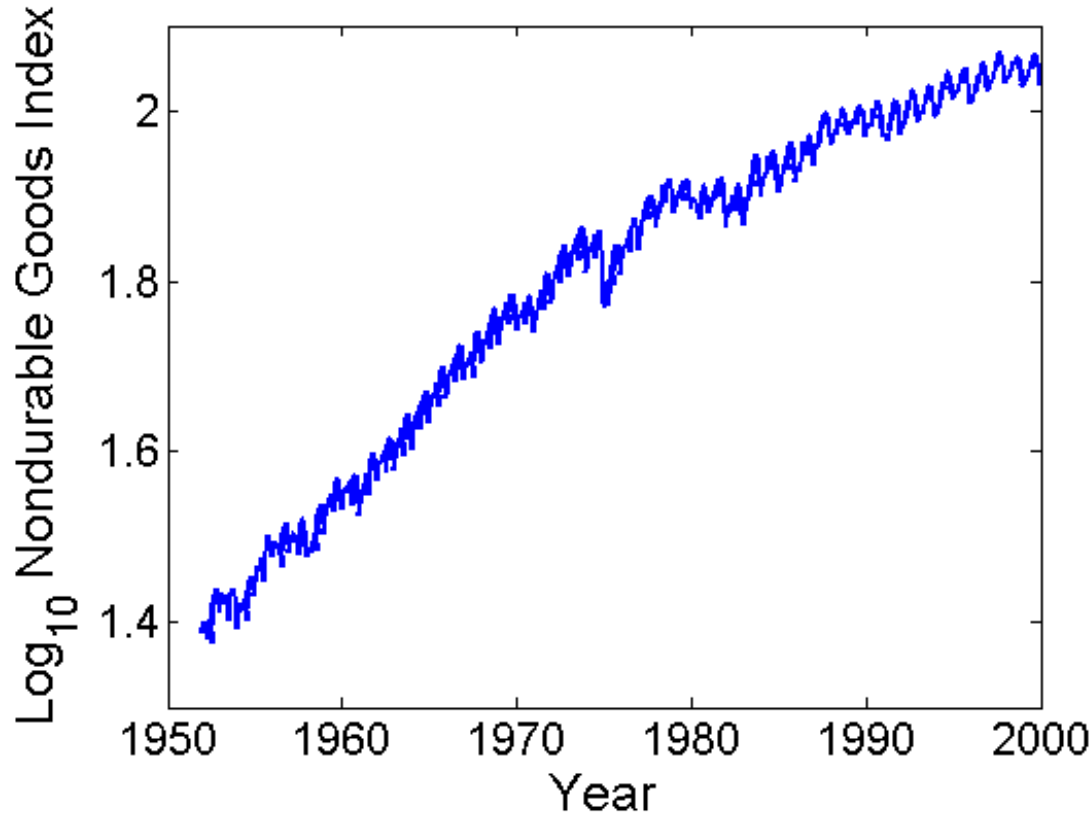
- The numerical integration in these equations is not as difficult as it seems. The scalar functions

$$\omega_{j\ell}(t) = \sum_i^N z_{ij}(t) z_{i\ell}(t)$$

play the role of *weighting functions* for the functional inner products

$$\int \theta_j(t) \theta'_\ell(t) \omega_{j\ell}(t) dt, j, \ell = 1, \dots, q.$$

3. Evolution in seasonal trend for the nondurable goods index



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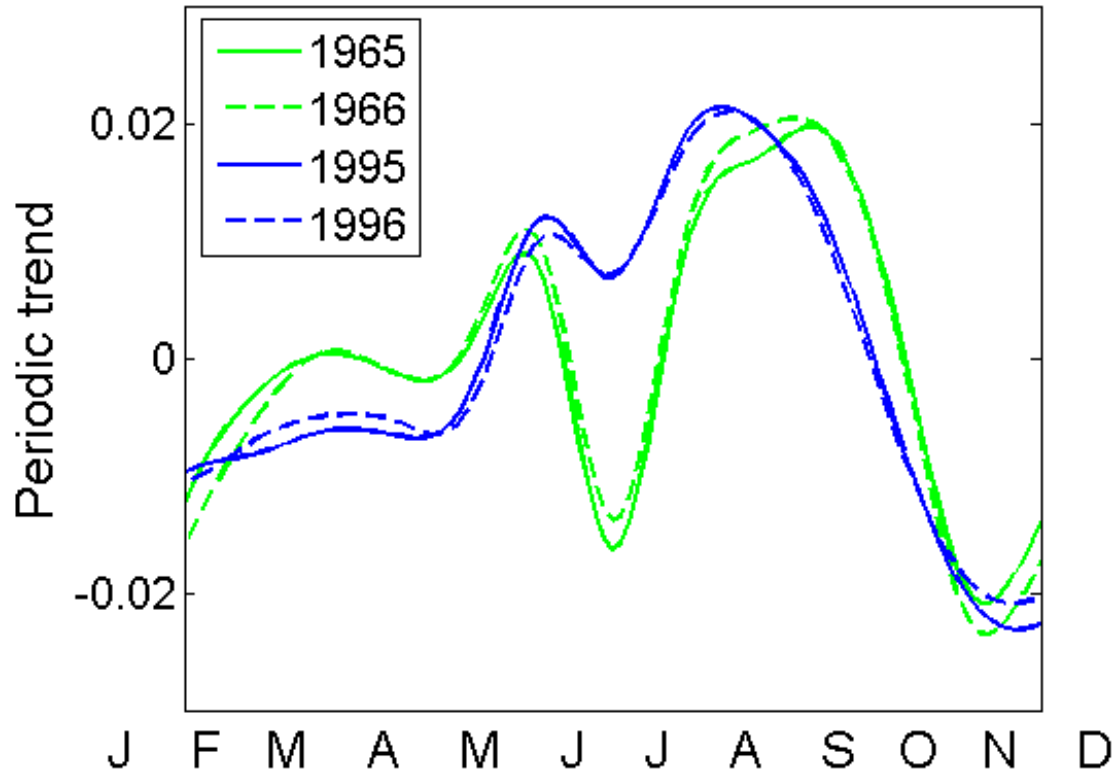
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Four seasonal trends



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- Seasonal trends are stable over a couple of years, but evolve over a longer time span.
- We can model nonseasonal trend plus an evolving seasonal trend as follows:

$$y(t) = \alpha(t) + \beta_1(t) \sin(2\pi t/365) + \beta_2(t) \cos(2\pi t/365) + \dots \\ + \beta_{p-1}(t) \sin(p\pi t/365) + \beta_p(t) \cos(p\pi t/365) + \epsilon(t)$$

- For the monthly index values from 1952 to 2000 we used $p = 10$.
- Intercept function α was modelled by B-splines with knots at each year and regularized with $\lambda = 0.01$.
- Each regression function β_j had 7 B-spline basis functions.
- A total of 121 parameters were estimated from 577 data points.

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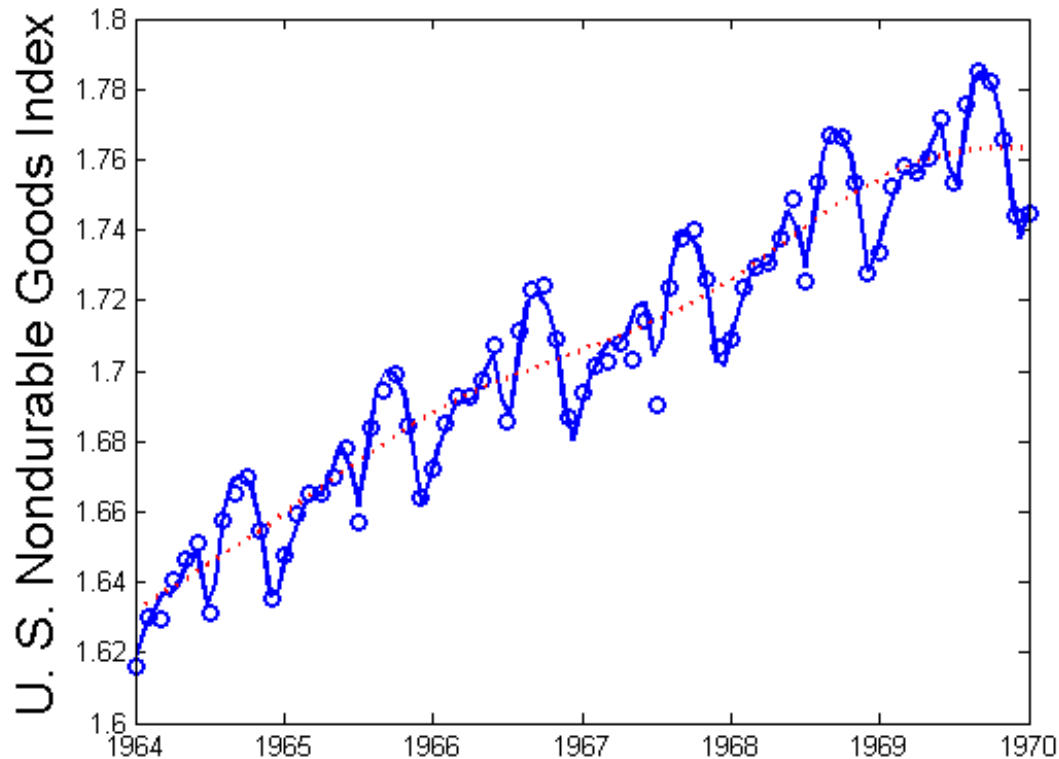
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The fit to the data over seven years



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4. Summary

- The concurrent functional linear model offers a simple way of relating a functional response to functional covariates.
- However, the influence is simultaneous, and does not permit a covariate to affect the outcome at any time other than the present.
- The model can also be fit to a single long time series provided that the number of parameters is kept small and/or regularization is used.

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