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EFFECTIVE STRING THEORIES AND FIELD THEORIES IN FOUR DIMENSIONS

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We argue that in four dimensions bosonic strings can be bundled together in a manner which resembles the way how individual filaments become bundled together to form a cable. The energy of such a bundle of strings is described by its extrinsic geometry, and it involves both torsion and curvature contributions. This leads to ordinary four-dimensional field theories, that describe the bundles of strings in terms of closed knotted solitons. Examples of field theories that can be constructed in this manner include the Faddeev model and the Skyrme model.

A closed bosonic string describes a circle that moves in a D-dimensional manifold. At very high energies the dynamics is governed by the Polyakov action. But at lower energies there can be corrections due to higher derivative terms that characterize the extrinsic geometry of the string. Particularly interesting is the extrinsic curvature contribution that relates to the second fundamental form of the string worldsheet.[1] Here we shall be interested in additional correction terms that are present when a bosonic string moves in three dimensions. We are mostly interested in the classical aspects of such an effective string theory. However, we suggest that the quantum theory can also be consistently formulated, even though we are not in 26 dimensions. This follows since the string theories we consider are effective theories that relate to four-dimensional quantum field theories which can be consistently quantized.

The embedding of a closed string $\mathcal{S}$ in a three-dimensional Euclidean space is uniquely determined by the Frenet equations, modulo rigid rotations and translations. These equations involve the curvature κ and torsion τ of the string, viewed as a curve in $\mathbb{R}^3$. In order to write the Frenet equations we describe the string by coordinates $\mathbf{x}(s) \in \mathbb{R}^3$ that we parametrize by the arclength $s \in [0, L]$ of the curve. The total length L of $\mathcal{S}$ can vary, and since the string is closed $\mathbf{x}(s + L) = \mathbf{x}(s)$. The unit tangent to $\mathbf{x}(s)$ is

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} \,. \tag{1}$$

The Frenet equations are

$$d\mathbf{t} = \frac{1}{2}\kappa(\mathbf{c}_+ + \mathbf{c}_-)ds\,, \tag{2}$$

$$d\mathbf{c}_\pm = -(\kappa\mathbf{t} \pm i\tau\mathbf{c}_\pm)ds\,. \tag{3}$$

Here $\mathbf{c}_\pm = \mathbf{n} \pm i\mathbf{b}$ and $\mathbf{n}$ is the unit normal vector and $\mathbf{b}$ is the unit binormal vector. The curvature κ and torsion τ are

$$\kappa = \frac{1}{2}\mathbf{c}_\pm \cdot \frac{d\mathbf{t}}{ds}\,, \tag{4}$$

$$\tau = \frac{i}{2}\mathbf{c}_- \cdot \frac{d\mathbf{c}_+}{ds}\,. \tag{5}$$

Both (4) and (5) are natural quantities for defining an energy functional for a string. For example, in classical continuum mechanics one often selects the energy of an elastic rod to contain at least the following three terms:

$$E = c_L L + \int_0^L ds(c_\kappa \kappa^2 + c_\tau \tau^2)\,. \tag{6}$$

Here the first term reflects the Polyakov action, in the static case. The second term relates to the extrinsic curvature of the string. It specifies the contribution introduced in Ref. 1 to the three-dimensional case. The last term measures the twisting of the string. To our knowledge this term has not really been discussed in connection to high energy string theories. But we find that it can be quite important when one relates effective string theories to field theories in $(3+1)$ dimensions. The $c_{L,\kappa,\tau}$ are some numerical parameters.

We are mainly interested in energy functionals like (6) that relate to four-dimensional field theories. For reasons that become apparent, from the present perspective the appropriate starting point is *not* an individual string but rather a collection of strings where the individual strings are bundled together in a smooth and well-groomed manner, much like individual filaments are bundled together to form a cable. For this we need to generalize (6) to an energy functional that describes the entire bundle in terms of its geometric quantities like extrinsic curvature and torsion. *A priori* there can be several distinct alternatives to bundle individual strings together. Here we shall employ a smooth three-component unit vector field $\mathbf{m}(\mathbf{x})$ which maps $\mathbb{R}^3$ to an internal two-sphere $\mathbf{S}^2_{\mathbf{m}}$. As we shall see this will indeed lead to a field theory description with a relativistic dynamics, in terms of geometric quantities that also relate to the torsion and curvature of the individual strings.

We are interested in configurations described by $\mathbf{m}$ that are localized in $\mathbb{R}^3$. This implies that at large distances $\mathbf{m}$ must approach a constant vector. Without any loss of generality we select the asymptotic vector to point along the negative z-axis, $\mathbf{m}(\mathbf{x}) \to -\mathbf{e}_3$ as $|\mathbf{x}| \to \infty$. The region where $\mathbf{m}$ deviates from the asymptotic value $-\mathbf{e}_3$ then describes our bundle of strings: The localized $\mathbf{m}$ is a mapping from the compactified three-space $\mathbb{R}^3 \sim \mathbf{S}^3$ to the internal $\mathbf{S}^2_{\mathbf{m}}$, and the pre-image of any point in $\mathbf{S}^2_{\mathbf{m}}$ corresponds to some individual closed string of the bundle in $\mathbb{R}^3$. These are the curves in $\mathbb{R}^3$ along which $\mathbf{m}(\mathbf{x})$ is a constant

$$\frac{d\mathbf{m}}{ds} = \frac{dx^i}{ds}\frac{\partial \mathbf{m}}{\partial x^i} \equiv t_i \partial_i \mathbf{m} = 0 \tag{7}$$

with t_i the unit tangent vector to the ensuing individual string $\mathcal{S} \sim \mathbf{x}(s)$ in the bundle.

In the general case the individual strings $\mathcal{S}$ are linked, so that the bundle forms a smooth and well-groomed knotted configuration in $\mathbb{R}^3$. Its self-linking number is computed by the Hopf invariant of $\mathbf{m}$ which coincides with the Gauss linking number for any pair of the closed strings $\mathcal{S}$. To compute the Hopf invariant we consider the curves along which $\mathbf{m}$ remains constant in $\mathbb{R}^3$. For this we introduce a four-component unit vector $\psi_\mu(\mathbf{x})$: $\mathbb{R}^3 \sim \mathbf{S}^3 \to \mathbf{S}^3$ that embeds the bundle in the configuration space $\mathbb{R}^3 \sim \mathbf{S}^3$, and define the combinations

$$z_1(\mathbf{x}) = \psi_1(\mathbf{x}) + i\psi_2(\mathbf{x}) \qquad \text{and} \qquad z_2(\mathbf{x}) = \psi_3(\mathbf{x}) + i\psi_4(\mathbf{x})\,. \tag{8}$$

We can select ψ_μ so that it yields the unit vector $\mathbf{m}$ by

$$\mathbf{m} = Z^\dagger \boldsymbol{\sigma} Z\,, \tag{9}$$

where $Z = (z_1, z_2)$. Notice that this does not define Z uniquely but there is a U(1) gauge symmetry, the relation (9) remains intact when we multiply Z by a phase

$$Z \to e^{\frac{i}{2}\xi} Z\,. \tag{10}$$

When we parametrize

$$Z = \begin{pmatrix} e^{i\phi_{12}} \sin\frac{1}{2}\vartheta \\ e^{i\phi_{34}} \cos\frac{1}{2}\vartheta \end{pmatrix} \tag{11}$$

and set $\beta = \phi_{34} - \phi_{12}$ and $\gamma = \pi - \vartheta$ we get

$$\mathbf{m} = \begin{pmatrix} \cos\beta \sin\gamma \\ \sin\beta\sin\gamma \\ \cos\gamma \end{pmatrix}\,. \tag{12}$$

Hence the bundle resides in the region in $\mathbb{R}^3$ where $\gamma \neq \pi$. Since $\mathbf{m}$ is independent of $\alpha = \phi_{12} + \phi_{34}$, we can identify $\alpha(\mathbf{x})$ as a coordinate generalization of the parameters s. Thus $\alpha(\mathbf{x})$ is the bundle coordinate that describes the individual strings $\mathcal{S}$ in the directions $\mathbf{t}$, the curves in $\mathbb{R}^3$ which are traced by constant values of $\mathbf{m}(\mathbf{x})$. The U(1) gauge transformation (10) sends $\alpha \to \alpha + \xi$ and relates to the reparametrizations $s \to \tilde{s}(s)$ of the individual strings. Since the physical properties of the bundle should be reparametrization invariant, any physical quantity should reflect an invariance under the gauge transformation (10).

We define

$$A_i = i(\partial_i Z^\dagger Z - Z^\dagger \partial_i Z)\,. \tag{13}$$

We substitute (11) into (13) and combine the individual terms as follows:

$$A = \cos\gamma d\beta + d\alpha\,. \tag{14}$$

The U(1) gauge transformation (10) sends A into

$$A \to \cos\gamma d\beta + d(\alpha + \xi) \tag{15}$$

which identifies A as the corresponding gauge field. Its exterior derivative yields the pull-back of the area two-form on $\mathbf{S}^2_{\mathbf{m}}$,

$$F = dA = \sin\gamma\, d\beta \wedge d\gamma = -\frac{1}{2}\mathbf{m}\cdot d\mathbf{m}\wedge d\mathbf{m} \tag{16}$$

and the dual one-form with components

$$B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$$

is parallel to the tangents of the individual strings $\mathbf{x}(s)$ since

$$\varepsilon_{ijk}t_j B_k = 0\,.$$

The Hopf invariant is

$$Q_H = \frac{1}{8\pi^2}\int F\wedge A = \frac{1}{8\pi^2}\int \sin\gamma\, d\alpha\wedge d\beta\wedge d\gamma\,. \tag{17}$$

If the Hopf invariant is nonvanishing, the bundle forms a nontrivial knot. In that case the flat connection $d\alpha$ cannot be entirely removed by the gauge transformation (15) since the Wilson loops

$$\oint_{\mathcal{S}} d\alpha = \int_0^L ds\, t_i\partial_i\alpha$$

along the closed strings $\mathcal{S}$ in the bundle are necessarily nontrivial.

In order to obtain appropriate generalizations of the curvature and torsion (2), (3) and in particular the ensuing version of (6) we consider a generic point $\mathbf{m}\in\mathbf{S}^2_{\mathbf{m}}$. Its pre-image corresponds to a generic string $\mathcal{S}$ in the bundle. We introduce the tangent vectors $\mathbf{e}_\pm = \mathbf{e}_1 \pm i\mathbf{e}_2$ of $\mathbf{S}^2_{\mathbf{m}}$ in $\mathbb{R}^3_{\mathbf{m}}$ so that $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{m})$ is a right-handed orthonormal triplet. When we parametrize $\mathbf{m}$ according to (12) the canonical choice is

$$\mathbf{e}_1 = \begin{pmatrix} \sin\beta \\ -\cos\beta \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} \cos\beta\cos\gamma \\ \sin\beta\cos\gamma \\ -\sin\gamma \end{pmatrix}. \tag{18}$$

These vectors describe a (small) neighborhood $\mathcal{M}$ around $\mathbf{m}$ in $\mathbf{S}^2_{\mathbf{m}}$. The pre-image of $\mathcal{M}$ under $\mathbf{m}(\mathbf{x})$ defines a (thin) tubular neighborhood $\mathcal{T}$ around $\mathcal{S}$ in $\mathbb{R}^3$. At each point $\mathbf{x}(s)$ along $\mathcal{S}$ we intersect the tube $\mathcal{T}$ by a disk-like surface $\mathcal{D}(s)$, normally to each of the strings that lies inside $\mathcal{T}$. For each s the surface $\mathcal{D}(s)$ then provides a cross-section of the tube $\mathcal{T}$. This is mapped onto $\mathcal{M}$ by the vector field $\mathbf{m}(\mathbf{x})$. In the limit of a very narrow tube $\mathcal{T}$ the discs $\mathcal{D}(s)$ become surfaces with an area element which is given by the pull-back of (16), with a unit normal which coincides with the pertinent vector $\mathbf{t}$.

In general the $\mathcal{D}(s)$ are curved in $\mathbb{R}^3$ and their curvature relates to the bending and twisting of the tube $\mathcal{T}$ in $\mathbb{R}^3$. A natural local measure of the curvature of the surface $\mathcal{D}(s)$ is given by the pull-back of the vector-valued one-form

$$d\mathbf{m} = -\partial_a \mathbf{x} B^a{}_b \, du^b \,, \tag{19}$$

where $B^a{}_b$ are the components of the curvature tensor on $\mathbf{S}^2_{\mathbf{m}} \in \mathbb{R}^3_{\mathbf{m}}$ and u^a are local coordinates in $\mathcal{M}$. According to (7), the components of (19) are indeed tangential to $\mathcal{D}(s)$. The projections of (19) along the tangent vectors of $\mathbf{S}^2_{\mathbf{m}}$

$$\Phi_i^{\pm} = \frac{1}{2}\mathbf{e}_{\pm} \cdot \partial_i \mathbf{m} = \frac{1}{2}(\pm i\partial_i \gamma - \sin\gamma \partial_i \beta) \tag{20}$$

then characterize the local curvature of $\mathcal{D}(s)$ along the directions in $\mathbb{R}^3$ that are determined by the pre-images of $\mathbf{e}_{\pm}$. Obviously (20) is a natural starting point for generalizing the curvature (4) to the bundle. But besides (10) we now also have a second U(1) gauge transformation. It acts in the internal $\mathbb{R}^3_{\mathbf{m}}$ by rotating the basis $\mathbf{e}_{\pm}$ according to

$$\mathbf{e}_{\pm} \to e^{\pm i\chi} \mathbf{e}_{\pm} \,. \tag{21}$$

Since any physical property of the bundle should be independent of a particular choice of basis vectors (18), the physical properties of the bundle should also reflect gauge invariance under (21).

The components in (20) are independent of the coordinate α, and hence manifestly invariant under the U(1) gauge transformations (10), i.e. reparametrizations. However, these components do not remain intact under the internal U(1) gauge transformations (21). Instead they transform according to

$$\mathbf{e}_{\pm} \cdot \partial_i \mathbf{m} \to e^{\pm i\chi} \mathbf{e}_{\pm} \cdot \partial_i \mathbf{m} \,.$$

But there is also an obvious, intimate relationship between rotating the points around each other in $\mathcal{M}$ and rotating the strings around each other in $\mathcal{T}$. This leads to a relation between the configuration space U(1) gauge transformations (10), i.e. reparametrizations of the strings, and the internal U(1) gauge transformations (21), i.e. rotations of the internal space basis vector. Consequently we redefine (20) into

$$\Phi_i^{\pm} \to \kappa_i^{\pm} = \frac{1}{2} e^{\pm i\alpha} \mathbf{e}_{\pm} \cdot \partial_i \mathbf{m} \,. \tag{22}$$

These $\kappa_i^{\pm}$ have a nontrivial weight under both U(1) gauge transformations. But they remain intact under the diagonal U(1) $\times$ U(1) gauge transformation where we compensate the internal U(1) rotation (21) by the configuration space reparametrization

$$\alpha \to \alpha - \chi \,. \tag{23}$$

We propose that (22) are the appropriate quantities that characterize the curvature of the bundle, for the purpose of constructing energy functionals.

In order to describe the torsion along the bundle we observe that besides (20), there is exactly one additional natural bilinear that can be constructed from the three unit vectors at our disposal

$$C_i = \frac{i}{2}\mathbf{e}_- \cdot \partial_i \mathbf{e}_+ = \cos\gamma \partial_i \beta \,. \tag{24}$$

A comparison with (5) suggests that C_i should indeed relate to the torsion along the bundle. However, we find that under the frame rotation (21) the C_i do not remain intact but transform like a U(1) gauge field

$$C_i = \frac{i}{2}\mathbf{e}_- \cdot \partial_i \mathbf{e}_+ \to \frac{i}{2}\mathbf{e}_- \cdot (\partial_i + i\partial_i \chi)\mathbf{e}_+ = C_i - \partial_i \chi \,.$$

Consequently we need to improve (24): Since the coordinate α relates to the parameters s along the individual strings, it becomes natural to employ the flat connection $d\alpha$ in (14) to generalize (24) into

$$C_i \to \tau_i = \frac{i}{2}\mathbf{e}_- \cdot (\partial_i + i\partial_i \alpha)\mathbf{e}_+ = \cos\gamma \partial_i \beta - \partial_i \alpha \,.$$

This should be compared with (14). The U(1) gauge transformation (21) now acts as follows:

$$\tau_i = \frac{i}{2}\mathbf{e}_- \cdot (\partial_i + i\partial_i \alpha)\mathbf{e}_+ \to \frac{i}{2}\mathbf{e}_- \cdot (\partial_i + i\partial_i[\alpha + \chi])\mathbf{e}_+$$

and τ_i remains invariant under the diagonal $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge transformation where we compensate (21) by the U(1) gauge transformation (10) that reparametrizes the bundle according to (23), much like in the case of (22). We propose that the τ_i are quantities that relate to the torsion of the bundle, in a natural generalization of (5) for the purpose of constructing energy functionals.

We note that the one-form with components τ_i has also been employed in the differential geometry of surfaces. There it is called the connection form[2] and can be shown to be a *globally* defined one-form on the unit tangent bundle of $\mathbf{S}^2$.

We also note that the components in $\kappa_i^\pm$ and τ_i are not independent but there are the following flatness relations between the ensuing one-forms

$$d\tau - 2i\kappa^+ \wedge \kappa^- = d\kappa^\pm - \tau \wedge \kappa^\pm = 0 \,. \tag{25}$$

These reduce the number of independent field degrees of freedom in $\kappa_i^\pm$, τ_i into three corresponding to the torsion, the curvature and the position along each of the individual strings.

We now proceed to construct an energy functional for the bundle. It turns out that the following has particularly interesting properties:

$$E = \mu^2 \int d^3x (\langle \kappa^+, \kappa^- \rangle + c|\tau|^2 + \lambda_\kappa \langle d\kappa^+, d\kappa^- \rangle + \lambda_\tau |d\tau|^2) \,. \tag{26}$$

Here μ has dimensions of mass, and c and $\lambda_{\kappa,\tau}$ are positive parameters. The first two terms in (26) are clearly analogs of the curvature and torsion terms in the Frenet energy (6). These two terms are also relevant operators in the ir limit, in the sense of renormalization group equations. The third and fourth terms in (26) are examples of marginal operators in the infrared limit; according to (25) these terms are both quartic polynomials in $\kappa^{\pm}$ and τ. There are also additional cubic and quartic polynomials in $\kappa^{\pm}$ and τ that could be added to the energy functional to produce a positive-definite energy functional. But here we limit ourselves to the four terms that are present in (26) since they have particularly interesting properties.

For example, if we set $c = \lambda_\kappa = 0$ the energy functional becomes independent of the flat connection $d\alpha$ and we arrive at the Faddeev model,[3]

$$E \to \mu^2 \int d^3x \left(\partial_i \mathbf{m} \cdot \partial_i \mathbf{m} + \frac{\lambda_\tau}{4} (\mathbf{m} \cdot d\mathbf{m} \wedge d\mathbf{m})^2 \right) . \tag{27}$$

This energy functional is bounded from below by the Hopf invariant (17),[4]

$$E \geq f \cdot |Q_H|^{3/4}$$

with f a nonvanishing number, and the model is known to support stable knotted solitons.[5,6] From the point of view of the present article, these solitons then admit an interpretation as knotted bundles of the closed strings $\mathcal{S}$.

Since each term in (26) is positive-definite the energy remains bounded from below by the Hopf invariant also in the more general case when all parameters are nonvanishing. Of particular interest is the case when $c = 1$ and $\lambda_\kappa = \lambda_\tau$: We recall the definition (8) of the unit four-vector ψ_μ which we employ to define the following SU(2) matrix:

$$U = \psi_4 + i\psi_a \sigma^a .$$

We can then combine the four terms in (26) into the following form:

$$E = \int d^3x \{ \mu^2 \cdot \mathrm{Tr}\, \partial_i U^{-1} \partial_i U + \lambda \cdot \mathrm{Tr}[U^{-1}\partial_i U, U^{-1}\partial_j U]^2 \} , \tag{28}$$

where we recognize the energy functional of the Skyrme model. Consequently from the present point of view the solitons of the Skyrme model are then knotted configurations in $\mathbf{m}$ but coupled to the connection $d\alpha$ and with a degenerate structure.[7]

Both the Faddeev and the Skyrme Hamiltonian lead to a relativistically covariant dynamics. The relation with the Skyrme model also tells that we can introduce statistics using the Wess–Zumino term.[8] This yields the important conclusion that the knotted solitons can be quantized either as bosons or as fermions.

For generic values of the parameters we expect the solitons of (26) to be knotted and qualitatively quite similar to those discussed in Refs. 5 and 6, except for degenerate cases such as the Skyrme model.[7]

Finally, we show how field theories lead to energy functionals of the form (6). For this we reduce the energy functional of the Faddeev model to that of an individual string in the bundle, by computing the leading contribution to the energy of the center-line of a knotted soliton: We describe an arbitrary point $\mathbf{X}$ in the bundle by

$$\mathbf{X} = \mathbf{x}(s) + r(\cos\theta \mathbf{n} + \sin\theta \mathbf{b}) \equiv \mathbf{x}(s) + r\mathbf{u} ,$$

where $\mathbf{x}(s)$ is a coordinate for the center-line and (r,θ) are plane polar coordinates in the plane through $\mathbf{x}(s)$, defined by the vectors $\mathbf{n}$ and $\mathbf{b}$. The angle θ is measured from the direction of $\mathbf{n}$ and s is the arclength measured from an arbitrary point on the center-line. When we move along $\mathbf{x}(s)$ by varying the parameter s, the vector $\mathbf{u}$ rotates around the center-line $\mathbf{x}(s)$. We define $\mathcal{N}$ to be the Gauss linking number between the center-line and a generic nearby curve of the form $\mathbf{x}(s)+\varepsilon\mathbf{u}(s)$ with some (small) constant ε. We define $g(s)$ to be an arbitrary function but with $g(L)=g(0)+L$, and introduce the angle variable

$$\phi = \theta + \frac{2\pi\mathcal{N}}{L}g(s)\,.$$

The mapping $s\to g(s)$ implements a reparametrization, while the use of ϕ as an independent variable instead of θ leads to a zero-framing of the curve $\mathbf{x}(s)$.[9] In the coordinates (r,ϕ,s) any point in a (thin) circular tube $\mathcal{T}$ with a (small) radius R surrounding the center-line can be described by

$$\mathbf{X} = \mathbf{x}(s) + r\cos\left[\phi - \frac{2\pi\mathcal{N}}{L}g(s)\right]\mathbf{n}(s) + r\sin\left[\phi - \frac{2\pi\mathcal{N}}{L}g(s)\right]\mathbf{b}(s)\,.$$

The components of the ensuing metric tensor are

$$d\mathbf{X}\cdot d\mathbf{X} = dr^2 + \left(1 - r\kappa\cos\left[\phi - \frac{2\pi\mathcal{N}}{L}g(s)\right] + r^2\hat{\tau}^2\right)ds^2 + r^2\,d\phi^2 + 2r^2\hat{\tau}\,d\phi\,ds\,,$$

where

$$\hat{\tau}(s) = \tau(s) - g'(s)\,.$$

We solve the equations of motion that follow from (27) for β and γ in (12) to the leading order r in the thin circular tube $R\to 0$ limit, and substitute the result to the action (27). When we select the function $g(s)$ so that it cancels an s-dependent contribution to γ, we find a static energy functional that involves the following universal terms:

$$E = aR^2\int ds|\partial_s\mathbf{x}|^2 + bR^4\int ds(3\tau^2(s) + R^2\cdot\kappa^2\tau^2) + \cdots\,. \qquad (29)$$

Observe that the energy does not contain a term which involves the curvature κ only, such as the middle term in (6). This is entirely consistent with the result that a stable knotted soliton must have a nontrivial self-linking number: A curvature contribution such as the middle term in (6) scales inversely in the length L of the string. Since the first term in (29) is proportional to L, the presence of such a curvature contribution would lead to a stable circular planar configuration with no self-linking.

In conclusion, we have studied relations between field theories and string theories. In particular, we have proposed that certain field theories can be viewed as effective theories of bundled strings, which then correspond to knotted solitons in the field theory model. We have constructed a general class of such field theory models by employing the natural geometric concepts of torsion and curvature of curves in $\mathbb{R}^3$. In the limit of thin bundles we recover the usual closed string action with additional torsion and curvature contributions. Since a field theory admits a natural particle interpretation, this leads to a curious dual picture between string-like excitations and pointlike particles. We hope that our results lead to a better understanding of such duality relations between strings and particles. Indeed, the investigation of these structures in the context of a Yang–Mills theory might provide new insight to the properties of colored flux tubes, and the appearance of a mass gap and color confinement in the theory.

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