Functional responses and functional covariates: The general case

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

Overview

- The fundamental question is this: If we have a covariate $z(s)$, how much of its variation over s should we use to fit the response $y(t)$ at fixed t?
- In the simplest case, we only use $z(t)$. We call this the *concurrent model* because it predicts y at time t by z at time t. We might call this "now-casting."
- If we want to use the behavior of z over an interval of values s, or over all values, things are more complex because this model has, effectively, an infinite amount of fitting power.

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

1. The full model for log precipitation

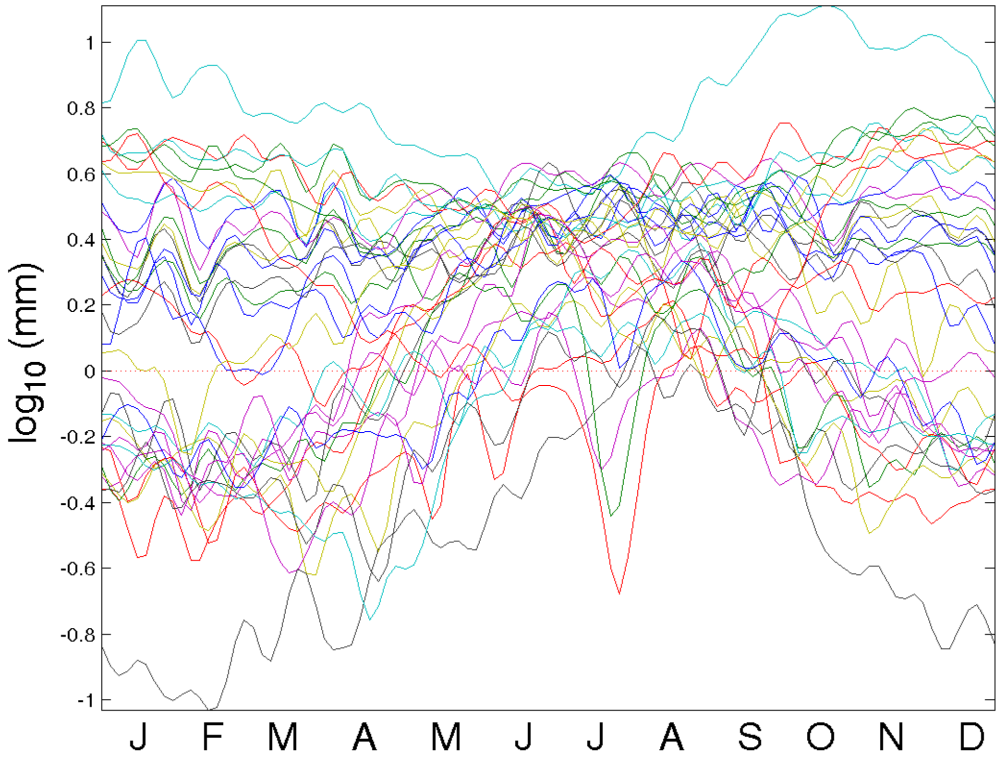
- We now want to predict the log precipitation profile $\texttt{LogPrec}_i(t)$ at time t from the entire temperature profile $\texttt{Temp}_i(s)$.
- The fitting criterion is

$$\texttt{LogPrec}_i(t) = \alpha(t) + \int_0^{365} \texttt{Temp}_i(s)\beta(s,t)\,ds + \epsilon_i(t) \ .$$

- $\beta(s,t)$ indicates the influence of temperature at time s on precipitation at time t.
- We can use the whole temperature profile because the data are periodic.
- We have already learned from predicting total log precipitation that we will have to apply a roughness penalty to $\beta(s,t)$ as a function of s.
- What about its variation as a function of t?

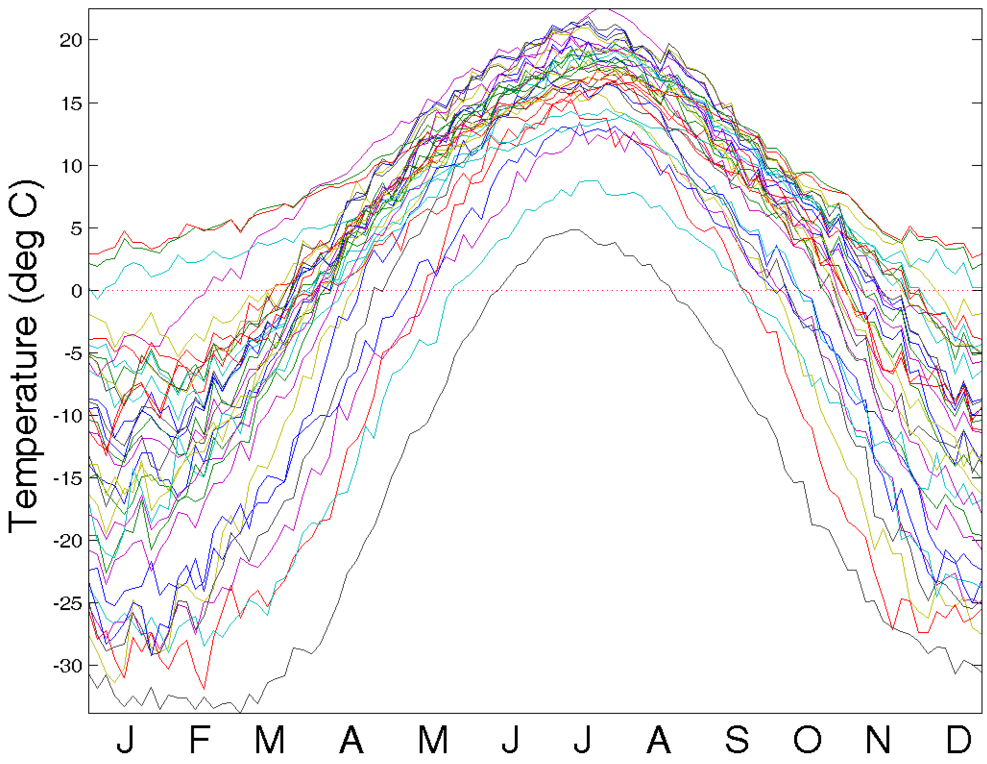
The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

Log precipitation functions



The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

Temperature functions

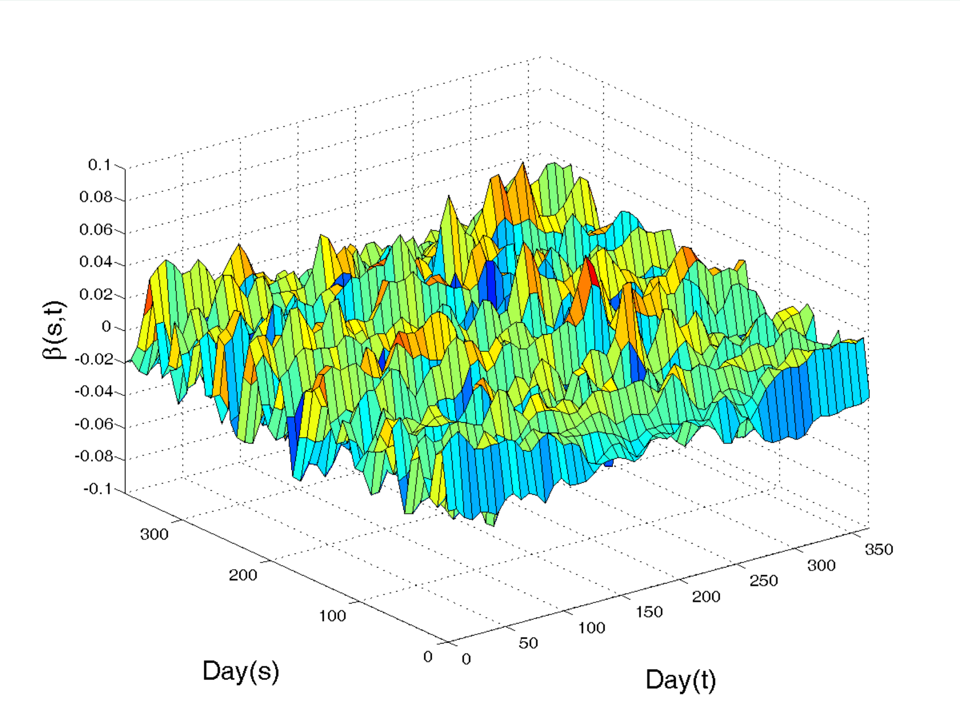


The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

- We apply two harmonic acceleration roughness penalties to $\beta(s,t)$, one for its variation in s, and one for its variation in t.
- Let's see what happens with fairly light penalties on both types of variation.
- We'll look at $\beta(s,t)$ and at the fit to the log precipitation data for Vancouver.

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

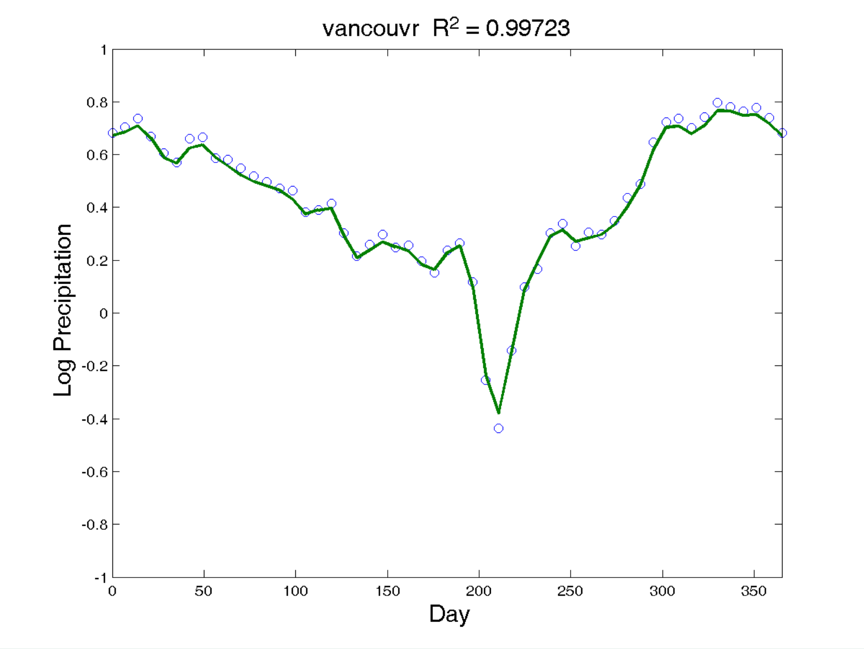
$\beta(s,t)$ has light penalties on s and t



- $\beta(s,t)$ is impossible to interpret.

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

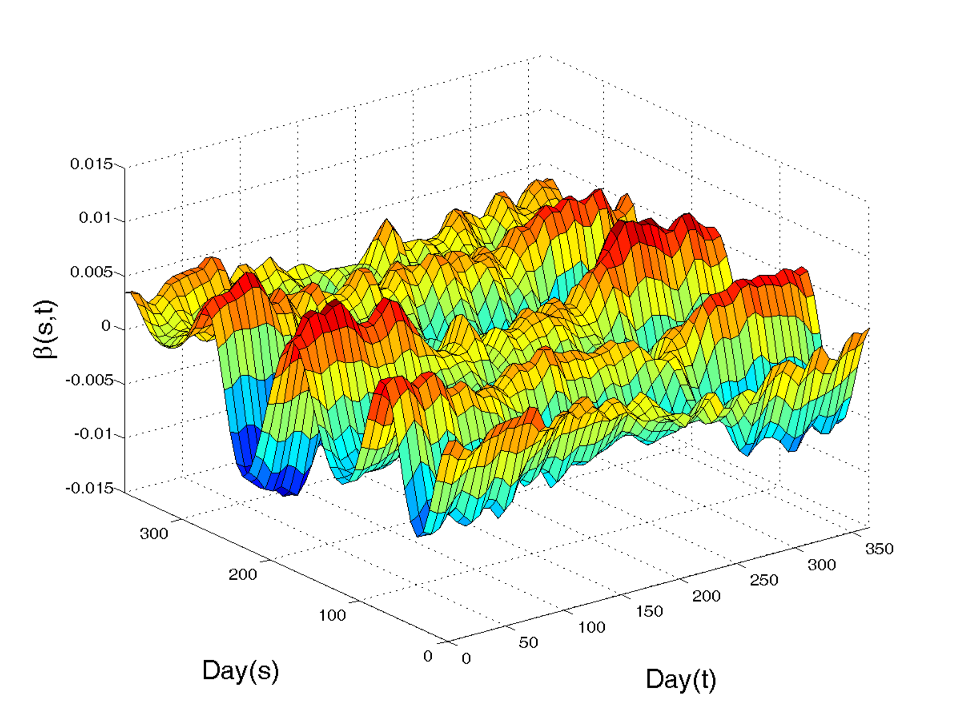
$\beta(s,t)$ has light penalties on s and t



- And we seem to have over–fitted Vancouver's data.
- Let's increase the smoothing parameter for s.

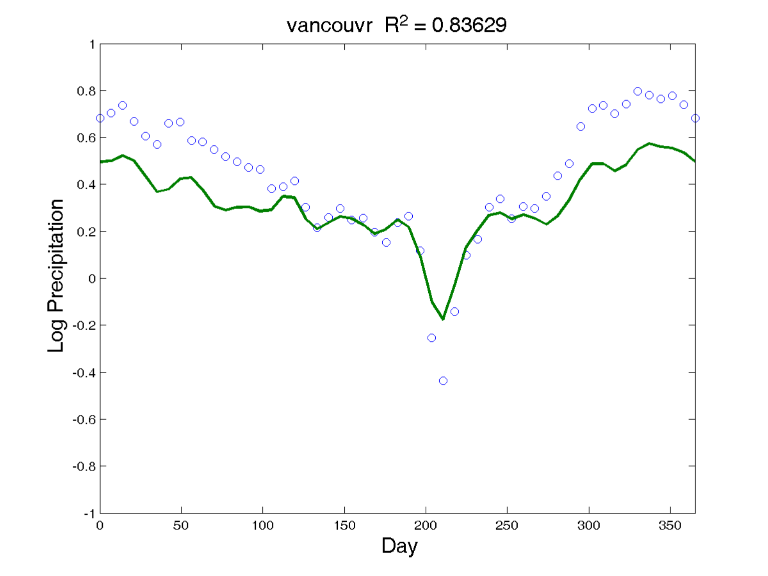
The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

$\beta(s,t)$ has heavy penalty on s and light on t



- $\beta(s,t)$ is interpretable as a function of s but impossible to understand in t.

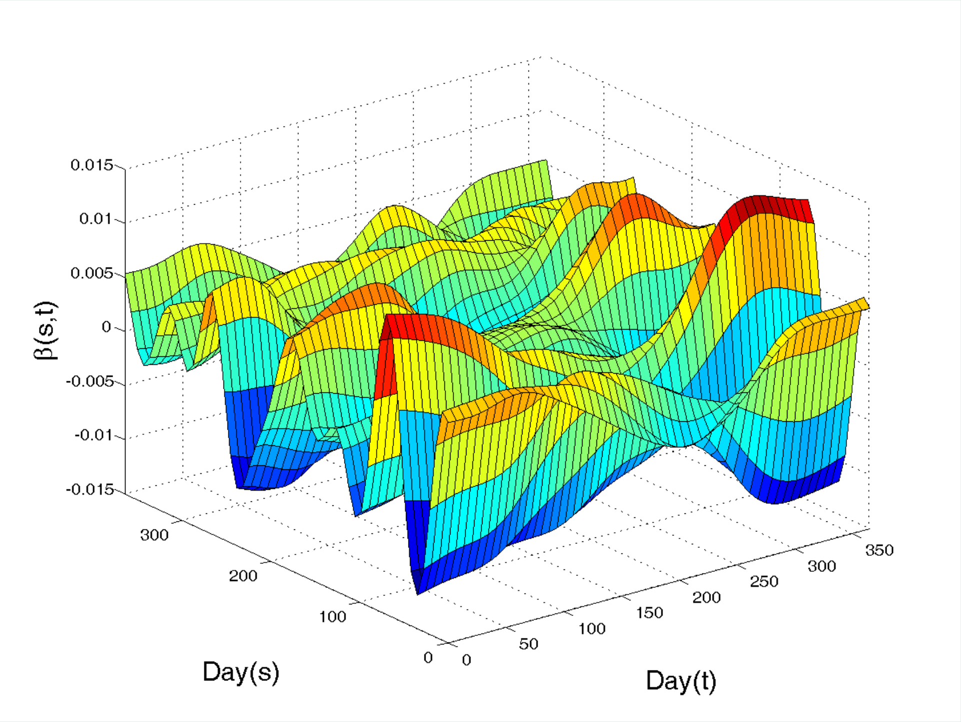
$\beta(s,t)$ has heavy penalty on s and light on t



- We now have a more reasonable fit to Vancouver's data, but the fitting function is too rough.
- Let's increase smoothing parameters for both s and t.

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

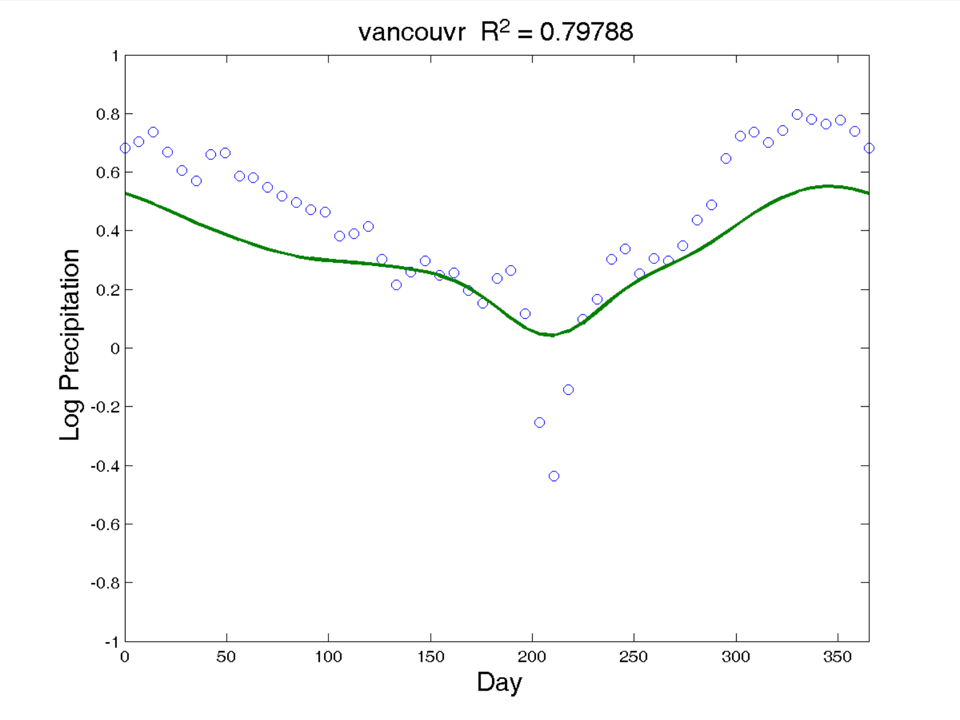
$\beta(s,t)$ has heavy penalties on both s and t



- $\beta(s,t)$ is now smooth in both s and t.

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

$\beta(s,t)$ has heavy penalties on both s and t



- The fit is reasonable and also smooth.

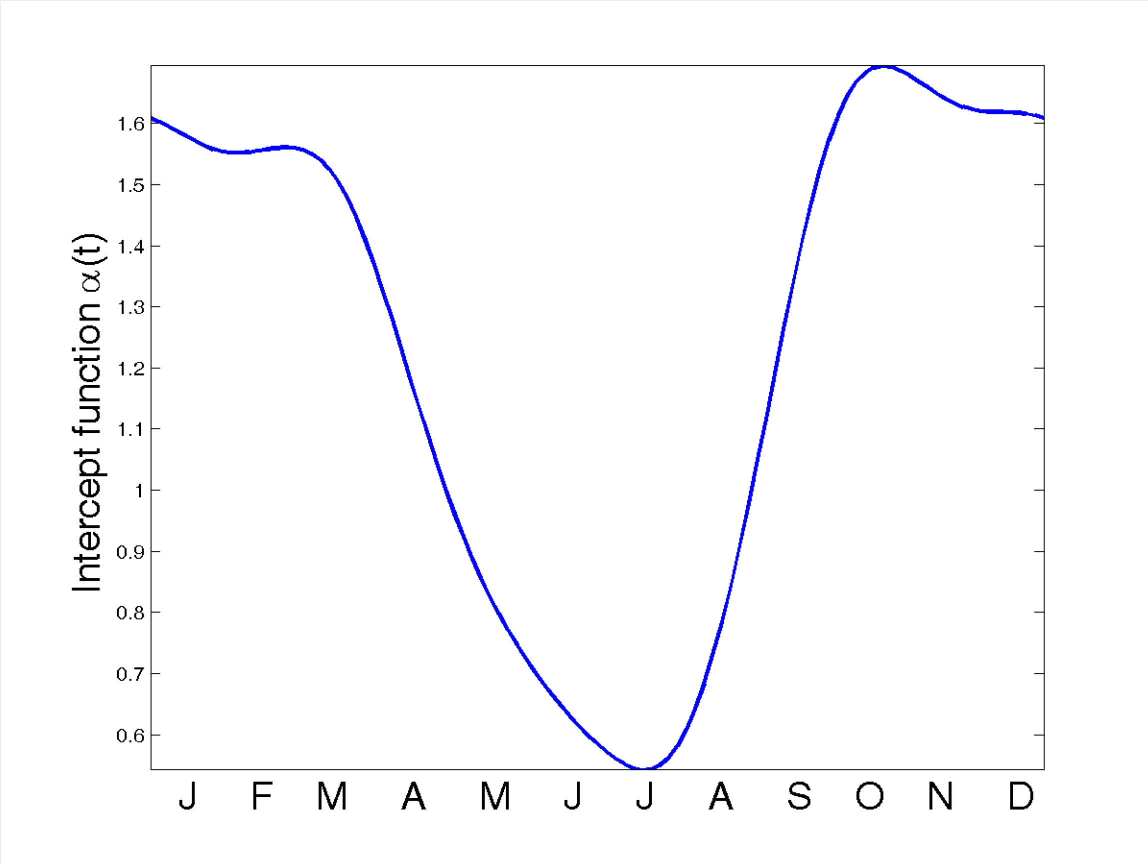
The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

What we see

- Penalizing the roughness of $\beta(s,t)$ as a function of s prevents over-fitting.
- Penalizing the roughness of $\beta(s,t)$ as a function of t allows us to see how the influence of temperature on precipitation varies from one time to another.
- We can now see that temperature is much more influential in the winter than in the summer.
- The rapid oscillation in s suggests that it is a derivative of temperature that really influences precipitation.

The intercept function $\alpha(t)$



The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

2. The historical model and other possibilities

- We were able to use all of $z(s)$ to predict $y(t)$ in the weather example because the data were periodic.
- In nonperiodic situations, it would only be meaningful to use the values of s *up to* t.

$$y(t) = \alpha(t) + \int_{t-\delta(t)}^{t} z(s)\beta(s,t)\,ds + \epsilon(s)$$

- For the lower limit of integration $t - \delta(t)$, the width $\delta(t)$ of the interval of integration can vary over t or can be constant.
- We can call this the *historical linear model*.
- See Ramsay and Silverman (2002) for an example.

The full model for log...
The historical model...
Lip acceleration...
Summary
Where to we go from here?

- More generally, we can integrate over a set Ω_t that varies with t,
- and also permit the covariate function z to vary over t as well,

$$y(t) = \alpha(t) + \int_{\Omega_t} z(s,t)\beta(s,t)\, ds + \epsilon(s)$$

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

- How do we construct basis function systems for complex regions of integration?
- The triangular mesh algorithms used in finite element methods to solve partial differential equations are natural here.
- Triangular meshes adapt well to nonstandard boundaries.
- Triangular basis functions also can be regularized.
- The PDEtools toolbox in Matlab contains powerful functions for mesh generation.

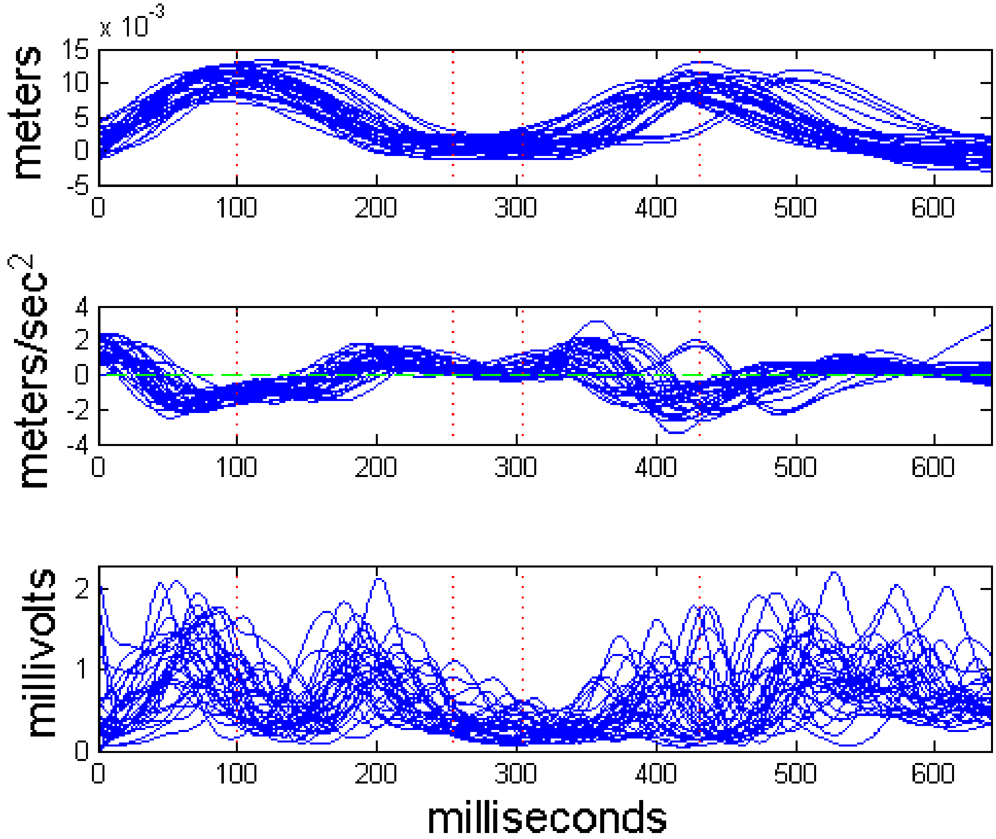
The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

3. Lip acceleration predicted from EMG signal

- Malfait and Ramsay (2004) studied how the acceleration of the lower lip while saying ”bob” was related to the EMG signal record from the depressor lip muscle. (see also Ramsay and Silverman, 2002)
- Each of 32 replications lasted for about 0.7 seconds.
- EMG is an indirect indication of muscle activation.
- It was hoped to learn something about the brain controls speech production.

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

32 replications of lip position and acceleration and of EMG signal



The feed–forward model

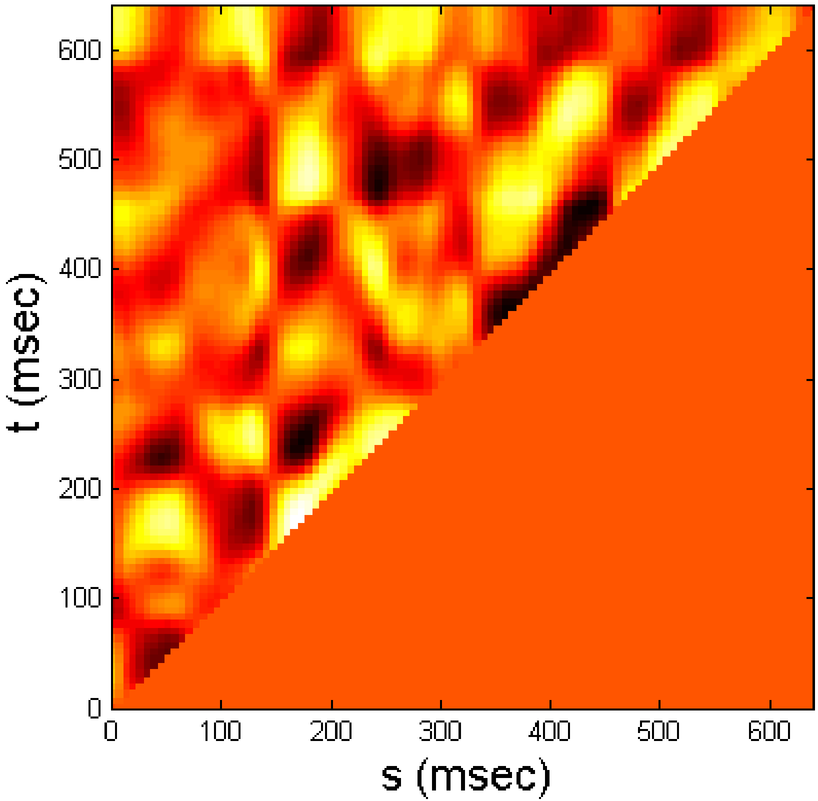
- Only feed–forward effects of EMG on acceleration are of interest.
- The model is

$$lip(t) = \alpha(t) + \int_{t-\delta}^{t} EMG(s)\beta(s,t)\,ds + \epsilon(s)$$

- Regression function $\beta(s,t)$ is defined over the triangular region $0 \leq s \leq t;\ 0 \leq t \leq 0.7$.
- How far back should we allow for an influence? How large should the lag δ be?

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

The correlation surface



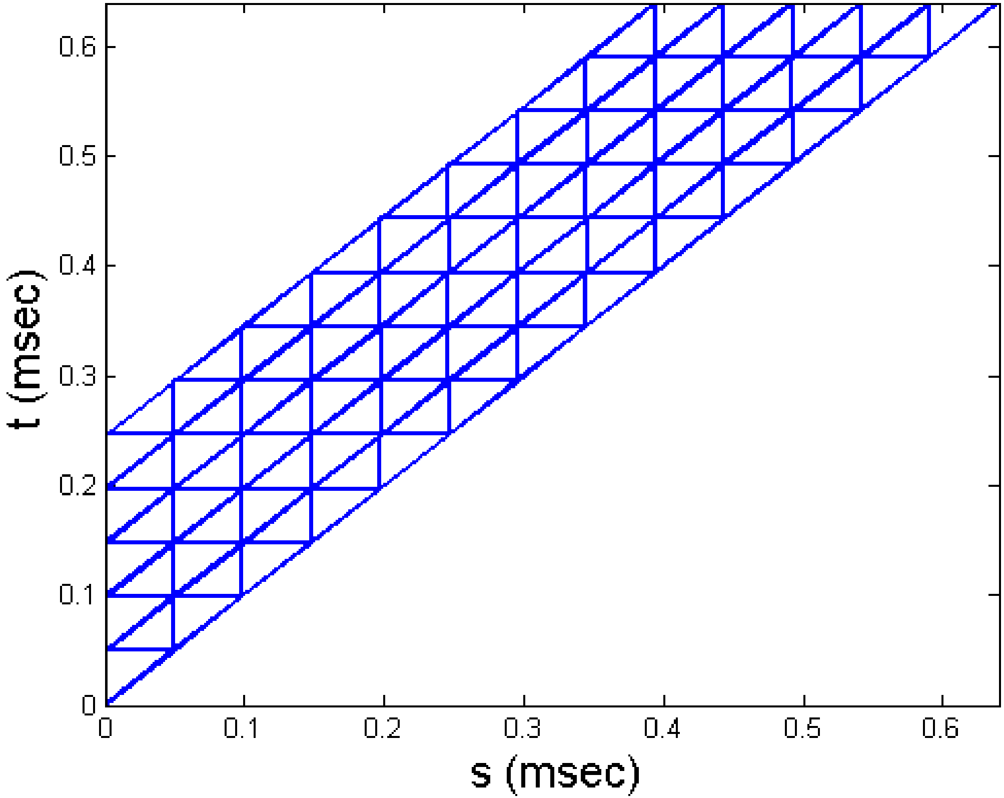
The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

The finite element basis for $\beta(s,t)$

- The finite element basis for functions defined over two arguments uses piece–wise linear functions defined over a triangular mesh.
- Triangular meshes can easily adapt to complex boundaries. Our problem here is particularly easy.
- The coefficient matrices become more and more sparse as the number of triangles increases.
- Sparse matrix computation, available in Matlab, makes for fast solutions.

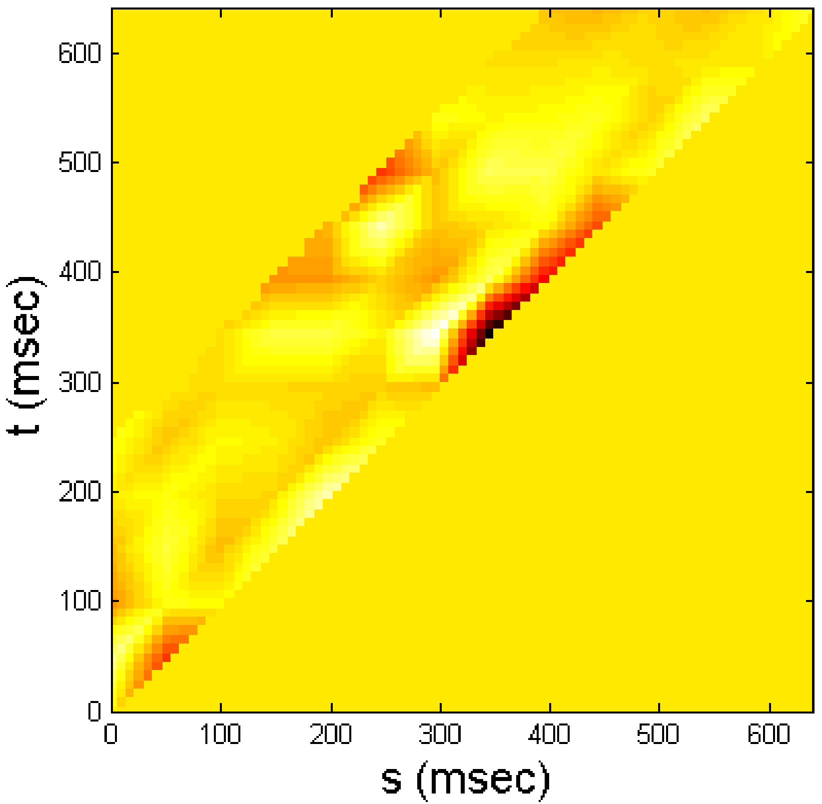
The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

A triangular mesh for the lip/EMG problem



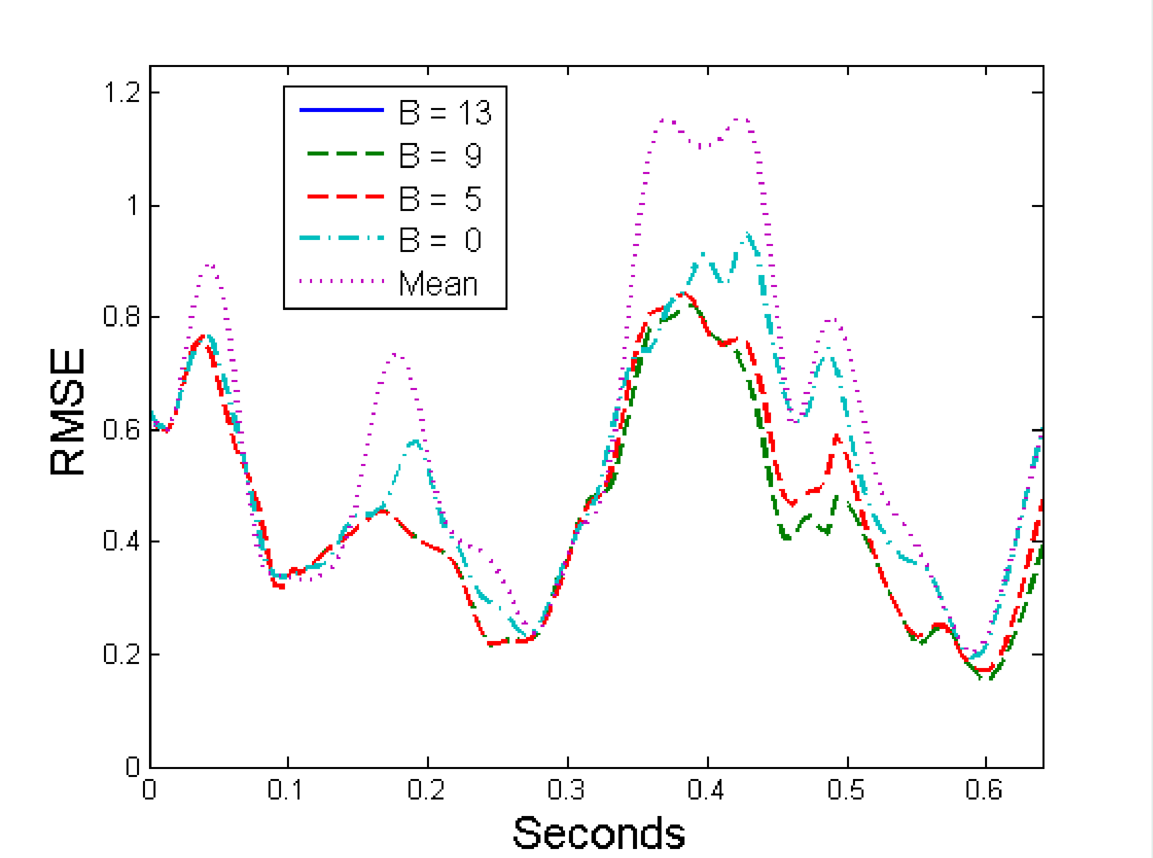
The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

The regression function surface for lag $\delta = 5$ triangles



The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

Standard error of estimate functions for various lags



The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

4. Summary

- When both response and covariates are functional, there are a lot of modelling possibilities.
- Using the full variation in a functional covariate $z(s)$ is only useful if we regularize the solution, either at the level of the regression coefficient $\beta(s,t)$, or at the level of the response $y(t)$.
- Using the full variation in the covariate usually only makes sense for periodic problems.
- A feed–forward model is more likely for nonperiodic functional regressions.
- Estimating the amount of functional history to use is an important issue.
- There is a lot more work to do in this exciting area!

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

The full model for log . . .
The historical model . . .
Lip acceleration . . .
Summary
Where to we go from here?

5. Where to we go from here?

- What if we want to use derivatives, $Dy(t)$ and $Dz(t)$, in our models?
- What about functional variables that interact with each other, such as we find in input/output systems with feedback?
- Nonlinear functional models?
- We have only tickled the ear of the elephant.