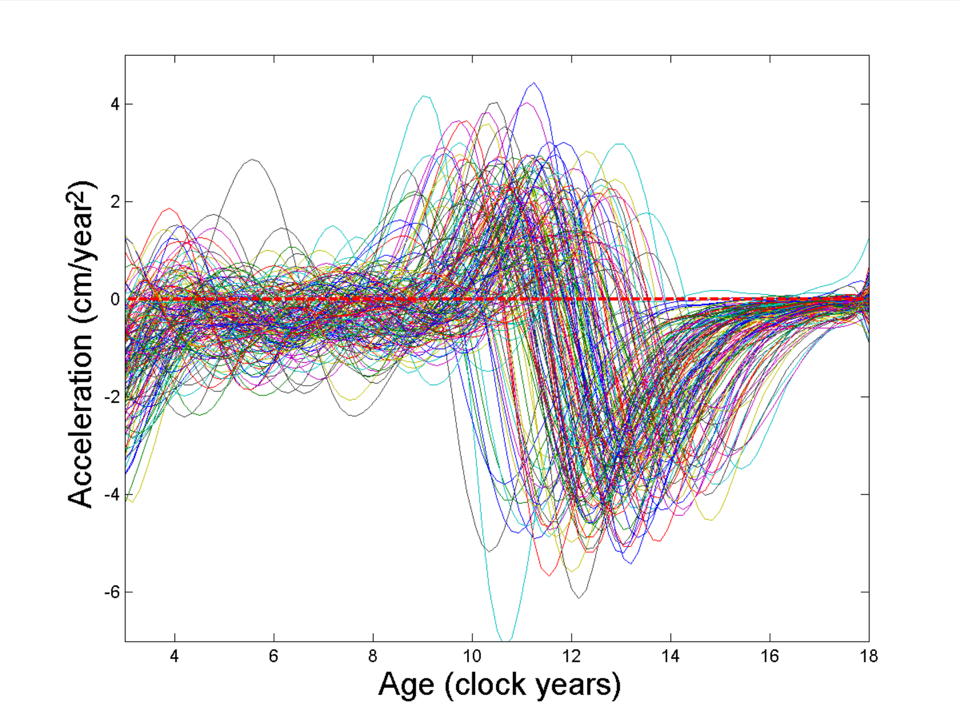
Human Growth: Separating Phase from Amplitude Variation

Alois Kneip, Gutenberg University, Mainz

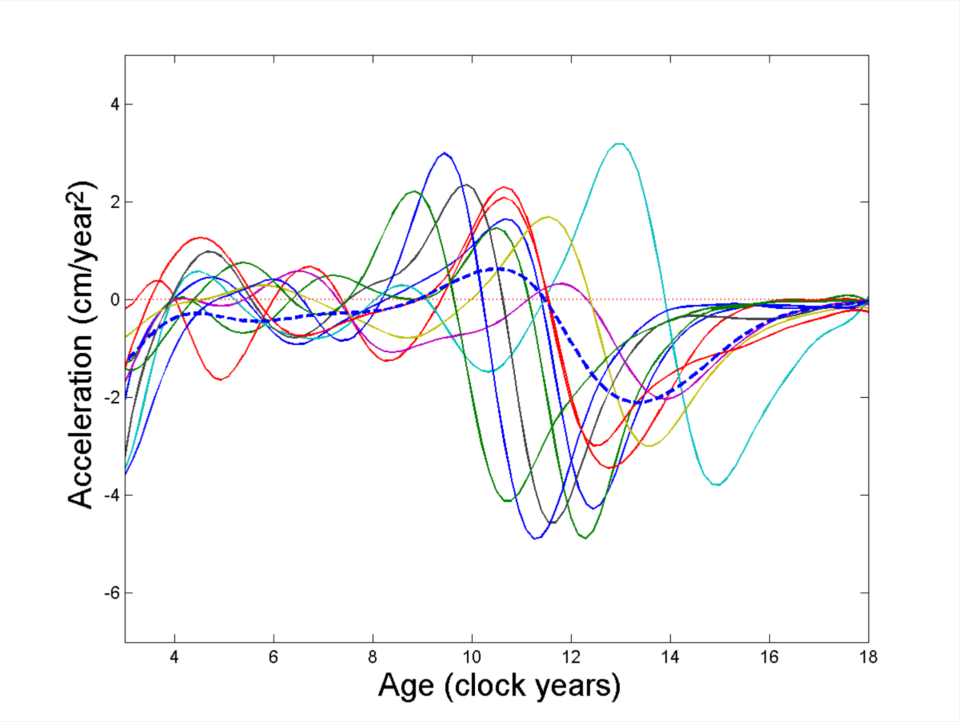
Jim Ramsay, McGill University

Accelerations $D^2h(t)$ for 158 girls



- The intensity of the pubertal growth spurt, as well as other aspects of acceleration, also varies. This is *amplitude variation*.
- The timing of puberty varies about twelve years by as much as two years. This is *phase variation*.
- Each girl has a biological clock that runs fast and/or slow relative to physical time. This is her own personal *system time*.
- How can we separate these two types of variation?
- We see why we need to do this in the next plot.

Accelerations $D^2h(t)$ for 10 girls and mean acceleration



The time warping function and registered growth

- Let $w(t)$ be a strictly increasing function that maps clock time t into biological time $w(t)$. Often called a *time-warping function*.
- Typically $w(0) = 0$ and $w(T) = t$ for some upper limit T.
- $$d(t) = w(t) - t$$

  is the *deformation function* that indicates whether the biological clock is running fast or slow, and $d(0) = d(T) = 0$.

- The *registered height function* defined by

$$h^*[w(t)] = (h^* \circ w)(t)$$

  contains only amplitude variation since growth is now measured relative to system time.

- The registered curve $h^*(t)$ is computed from $w(t)$ by
  - computing $w^{-1}(t)$ by smoothing t with respect to $w(t)$
  - smoothing $h(t)$ with respect to $w^{-1}(t)$

A model for amplitude and phase variation

- Alois proposed to combine time–warping with principal components analysis in this way, expressed for girl i:

$$h_i^*[w_i(t)] \approx \sum_r^p a_{ir} g_r(t)$$

- The principal component functions $g_r(t)$ reflect p modes of amplitude variation.
- The principal component scores a_{ir} mix these components together to fit girl i's registered curve.

An operator for separating amplitude and phase variation

- For relatively small amounts of deformation $d(t)$, we have the first order approximation

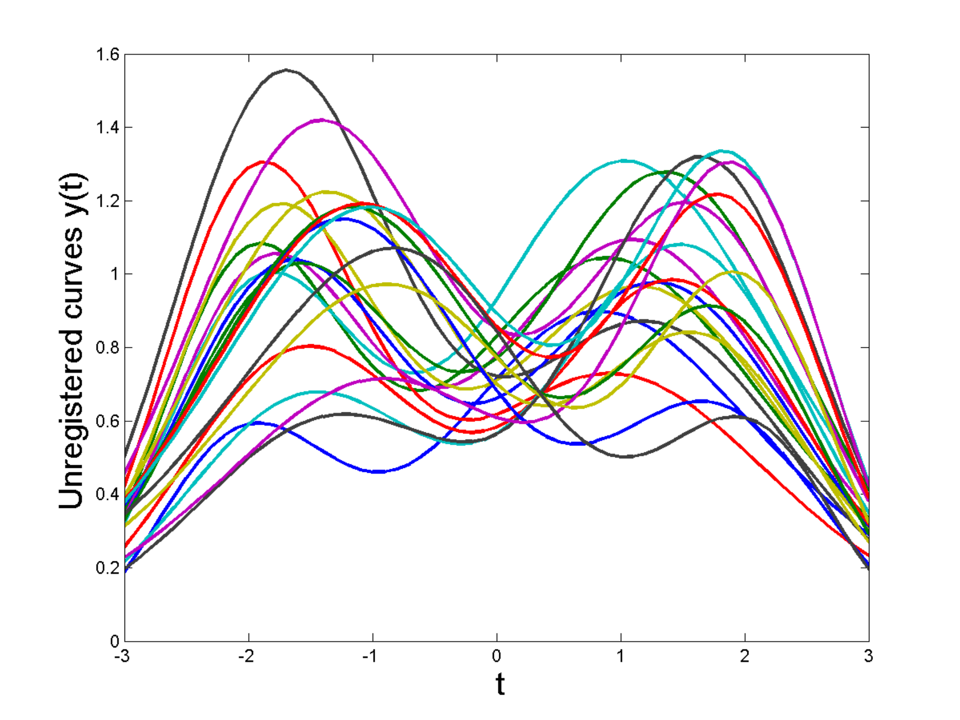
$$h_i[w_i(t)] \approx \beta_i h_i(t) + Dh_i(t) d_i(t)$$

- We can keep $d(t)$ small by applying a roughness penalty to d.
- Estimating $d_i(t)$ and β_i is a matter of using functional regression analysis.

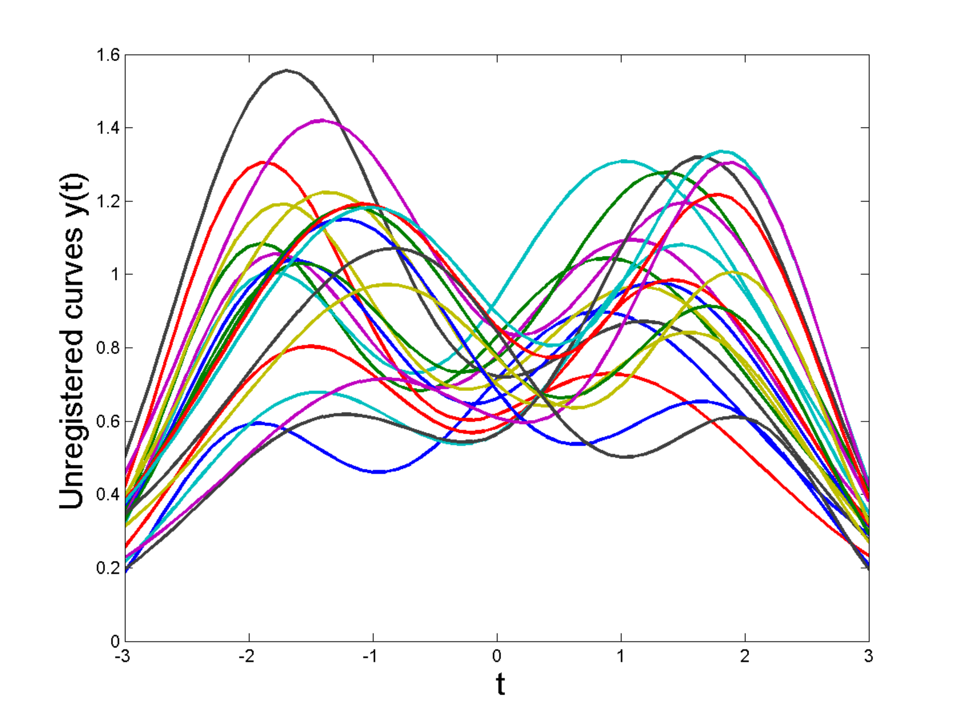
The PCA–with–warping algorithm

- Initially, we
  - compute a single principal component without centering the $h_i(t)$'s
  - compute the $d_i(t)$'s using the principal component as the registered target curve, keeping them small
  - register the curves using these $d_i(t)$'s.
- Recomputing the first principal component, we then subtract its approximation to each curve from the curve, and compute a single component of these residuals. Followed by the registration step.
- And so on up to p components.
- This entire cycle is repeated until convergence is achieved.

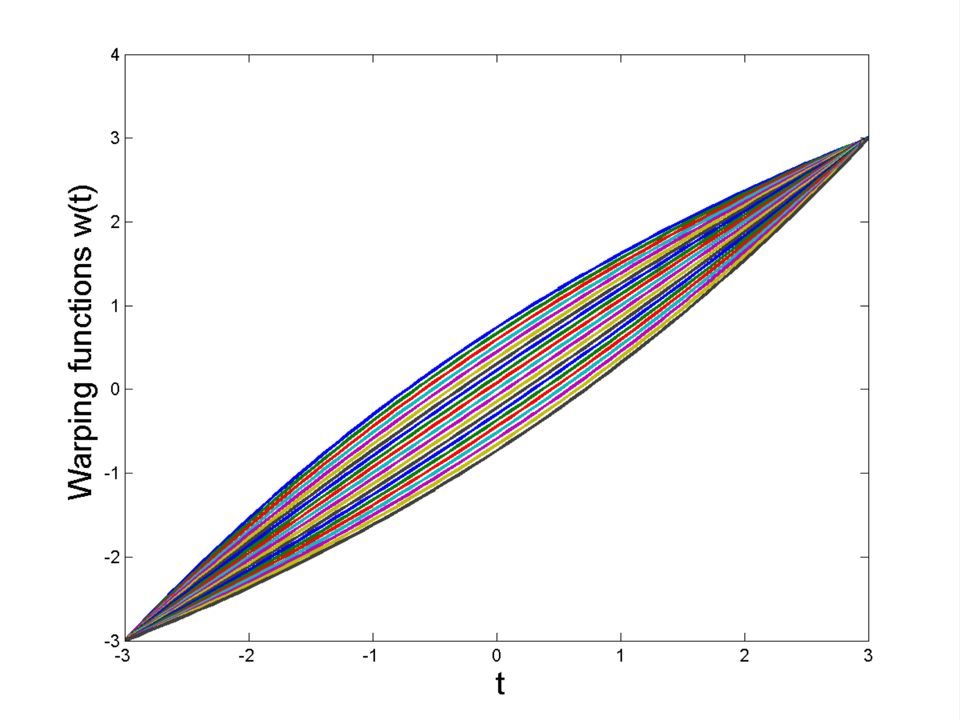
Some simulated curves having two components



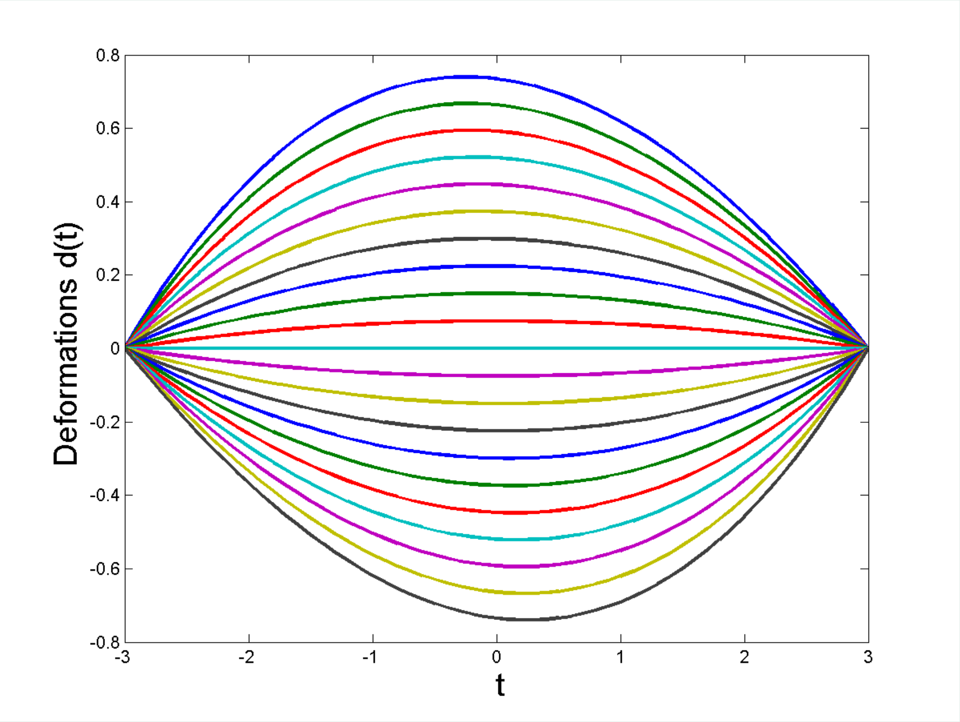
Some simulated curves having two components



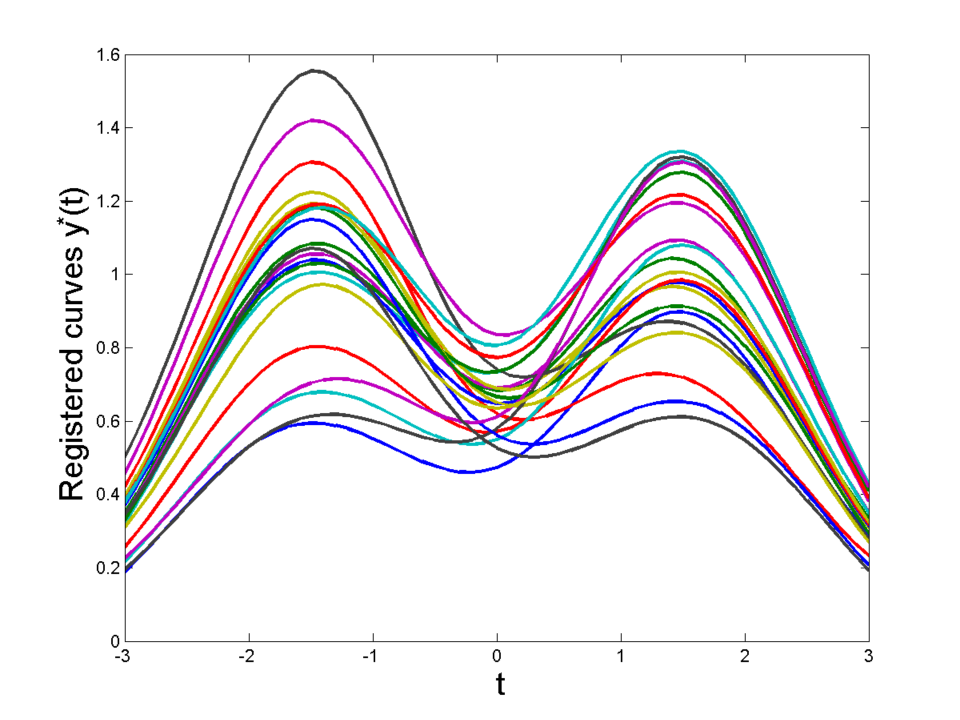
The warping functions $w_i t)$ to be estimated



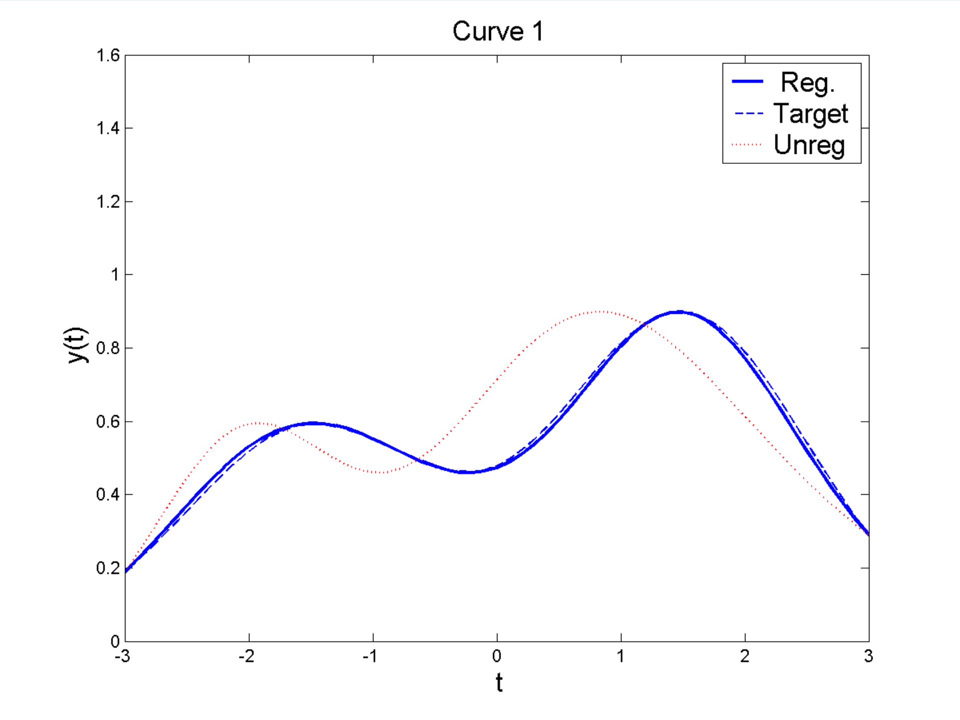
The deformation functions $d_i(t)$



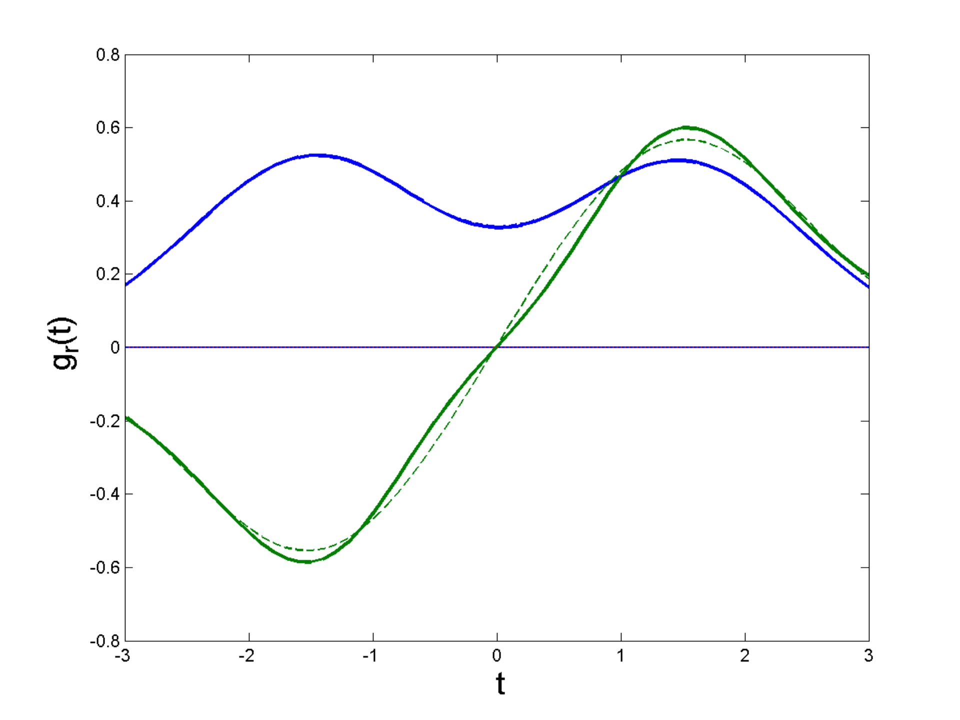
The registered curves



The first curve and its true value

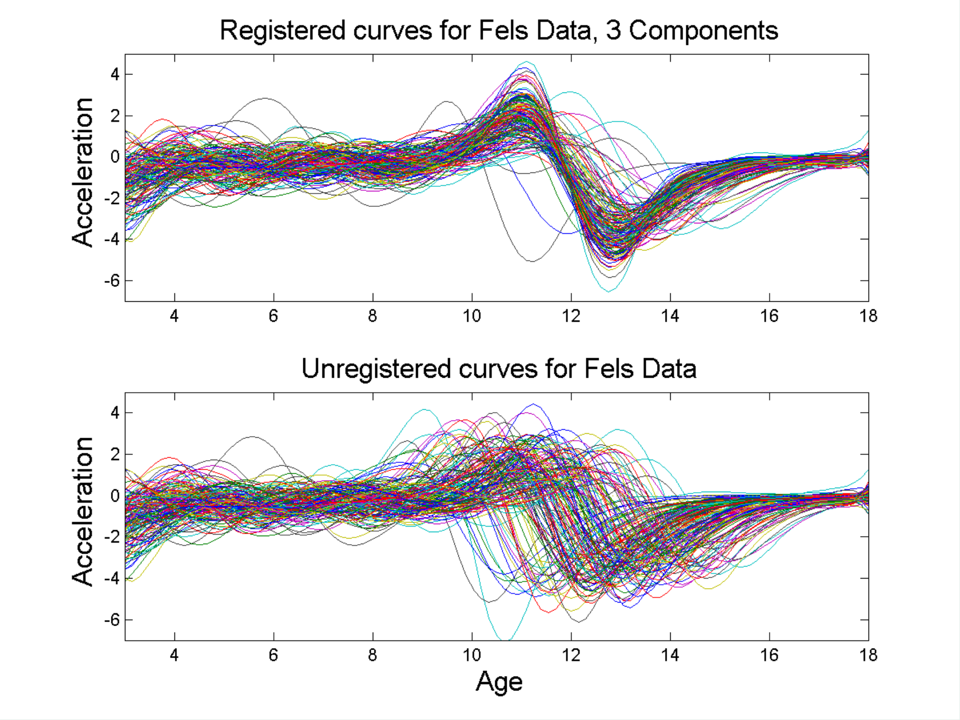


The principal components

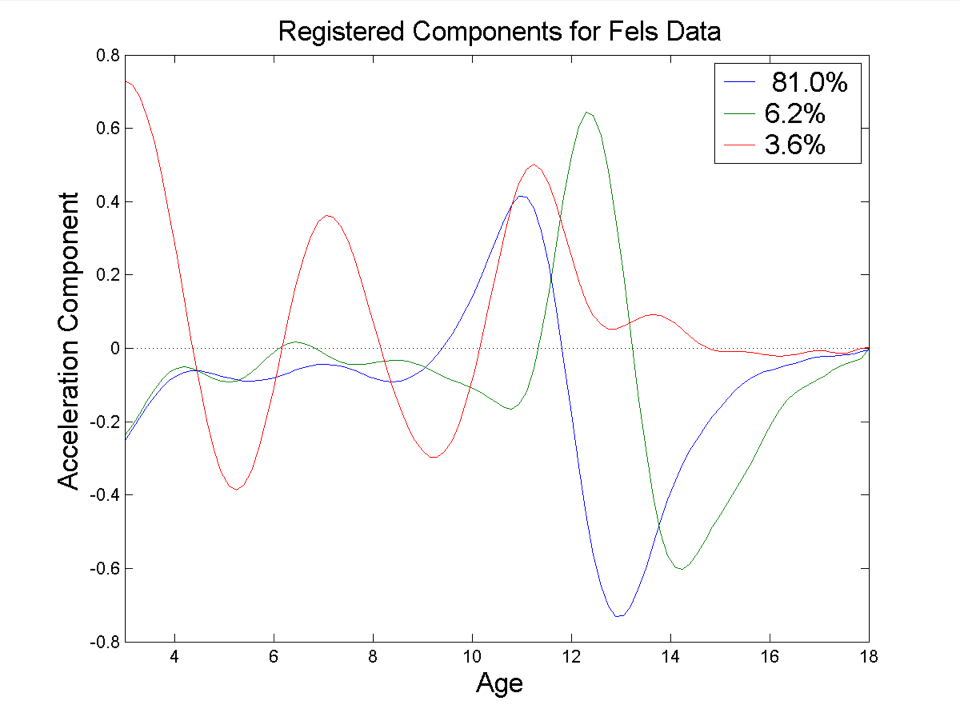


Analyzing the Fels accelerations

- We opted for $p = 3$ principal components of variation for the registered curves.
- Before registration, three principal components account for 84% of the variation in the data.
- After registering to the estimated three principal components, 91% of the purely amplitude variation is accounted for.



Registered curves for Fels Data, 3 Components
Acceleration
4
2
0
-2
-4
-6
4
6
8
10
12
14
16
18
Unregistered curves for Fels Data
Acceleration
4
2
0
-2
-4
-6
4
6
8
10
12
14
16
18
Age


Registered Components for Fels Data
81.0%
6.2%
3.6%
Acceleration Component
Age
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
4
6
8
10
12
14
16
18

Conclusions

- Growth takes places on short, medium, and long time scales.
- A large part of the variation in growth curves is due to phase, both as the timing of the pubertal growth spurt, and also as time shifts in smaller growth spurts before puberty.
- We have methods for separating phase from amplitude variation.
- This makes it possible to obtain better summaries of the components of amplitude variation.
- Phase variation can then be studied separately, and also linked by cross-correlational analyses to amplitude variation.