



Vortices and flat connections

Sazzad Mahmud Nasir ¹

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

Received 17 October 1997

Editor: P.V. Landshoff

Abstract

At Bradlow's limit, the moduli space of Bogomol'nyi vortices on a compact Riemann surface of genus g is determined. The Kähler form, and the volume of the moduli space is then computed. These results are compared with the corresponding results previously obtained for a general vortex moduli space. © 1998 Elsevier Science B.V.

1. The Abelian Higgs model in $(2 + 1)$ dimensions is an interesting arena to study vortices. The coupling constant of the model determines the nature of interactions among vortices. At the critical coupling, the model admits static and finite energy Bogomol'nyi vortex solutions [1]. Stability of these solutions is ensured by topology. We will consider vortices in a space-time of the form $\mathbf{R} \times M$, where M is a compact two dimensional manifold. The metric of the space-time is taken to be $ds^2 = dx_0^2 - \Omega(x_1, x_2)(dx_1^2 + dx_2^2)$, where x_1 and x_2 denote local coordinates on M . Let A_μ , ($\mu = 0, 1, 2$) be a $U(1)$ gauge potential and ϕ be a complex scalar field.

Working in the gauge $A_0 = 0$, the Lagrangian of the model at the critical coupling is $L = T - V$, where

$$T = \frac{1}{2} \int_M d^2x \left(\dot{A}_i \dot{A}_i + \Omega \dot{\phi} \dot{\phi} \right), \quad i = 1, 2 \quad (1)$$

$$V = \frac{1}{2} \int_M d^2x \Omega \left[\frac{1}{2} F_{ij} F^{ij} + D_i \phi \overline{D_i \phi} + \frac{1}{4} (|\phi|^2 - 1)^2 \right] \quad (2)$$

are respectively, the kinetic and the potential energies. Here, $D_i = \partial_i - iA_i$, and $F_{12} = \partial_1 A_2 - \partial_2 A_1$, is the magnetic field. The following first order Bogomol'nyi vortex equations are obtained by minimizing the potential energy

$$(D_1 + iD_2) \phi = 0 \quad (3)$$

$$F_{12} + \frac{\Omega}{2} (|\phi|^2 - 1) = 0. \quad (4)$$

The above equations admit static multi-vortex solutions. The solutions are parametrized by a $2N$ dimensional moduli space, M_N , where N is the num-

¹ E-mail address: S.M.Nasir@damtp.cam.ac.uk.

ber of zeros of the Higgs field counted with multiplicity [2,3]. N – called the vortex number – is related to the total magnetic flux by $N = \frac{1}{2\pi} \int_M d^2x F_{12}$. The potential energy of a configuration of static N -vortices is πN . Topologically, M_N is just the symmetrized N -th power of M . The moduli space has a natural Riemannian metric induced from the kinetic energy expression (1). This is given by

$$ds^2 = \frac{1}{\pi} \int_M d^2x (\delta A_i \delta A_i + \Omega \delta \phi \delta \bar{\phi}), \quad i = 1, 2. \quad (5)$$

In obtaining (5) from (1) we have multiplied (1) by $2/\pi$ to agree with the conventions of [4]. It is to be noted that there is a Gauss' law constraint arising from the equation of motion of A_0 . This constraint ensures that M_N consists of the solutions of Eqs. (3), (4) modulo gauge transformations connected to the identity. Hereafter, by 'gauge transformations' we will mean those gauge transformations connected to the identity. Through Manton's work [5] it is known that at low energy – when most of the degrees of freedom remain unexcited – the moduli space can be used to describe interesting physical phenomena associated with vortices, such as scattering [6], thermodynamics [4,7,8], and the phase transition of vortices at near-critical coupling [9], etc. In the moduli space approximation the vortex dynamics can be thought of as geodesic motion on the moduli space.

For a compact surface, in order for vortex solutions to exist one has to satisfy Bradlow's bound [10]. The bound is $4\pi N \leq A$, where A is the area of M . This can be obtained by integrating (4) over M , and noticing that the integral of $|\phi|^2$ over M is positive. The bound means that for a given area there is a limit on the number of vortices one can have on M . At Bradlow's limit, $A = 4\pi N$, the Higgs field ϕ must vanish everywhere on M . Then, the Bogomol'nyi equations reduce to the following single equation

$$F_{12} = \frac{\Omega}{2}. \quad (6)$$

This is an equation for a constant magnetic field on M . The energy of the configuration is still πN .

It might be thought that (6) has a unique solution up to gauge equivalence, which in turn will mean

that the moduli space is just a point. However, if M has non-contractible loops (one-cycles), i.e. if the first homotopy group of M is non-trivial, then the moduli space of solutions of (6) is non-trivial as well. A solution of (6) up to gauge equivalence is given, in addition to a gauge potential that solves (6), by specifying holonomies around a basis of one-cycles. The holonomies around any two homologous loops are generally different as the magnetic field is non-zero on M , but the difference can be completely determined by using Stokes' theorem. Now, linearizing (6) around a particular solution one can see that the perturbed gauge potentials satisfy the equations for flat $U(1)$ connections for which the magnetic field is zero. Flat connections are associated with large gauge transformations. Flat connections up to gauge equivalence are also given by specifying their holonomies around a basis of one-cycles. Hence, when $\phi = 0$ the moduli space of Bogomol'nyi vortices is no longer M_N . Locally, the tangent space of the vortex moduli space at Bradlow's limit can be identified with the tangent space of the space of flat $U(1)$ connections on M . With an abuse of notation, in future the moduli space of solutions of (6) and the space of flat $U(1)$ connections will be denoted by M_f . It was demonstrated long ago by Aharonov and Bohm [11] that flat connections play quite a non-trivial role in quantum physics. It should be noted that flat connections do not contribute to the moduli of the Bogomol'nyi equations when the Higgs field is non-vanishing.

In passing we would like to point out that when $\phi = 0$, one may consider $F_{12} = 0$ as solutions of the static Abelian Higgs model. The energy is then $A/8$ with no restriction on the value of A . However, these solutions cannot be obtained from Bogomol'nyi equations. Henceforth, we will only consider vortices at Bradlow's limit, and M is taken to be of genus $g \geq 1$.

2. Let $A = A_1 dx_1 + A_2 dx_2$ solve (6). Then, $A + \delta A$, where $\delta A = a_1 dx_1 + a_2 dx_2$ is a flat connection, is also a solution of (6). The equation satisfied by δA is $d\delta A = 0$. The metric on M_f is obtained by putting $\phi = 0$ in (5). This is

$$ds^2 = \frac{1}{\pi} \int_M d^2x [(a_1)^2 + (a_2)^2]. \quad (7)$$

Notice that the metric of M does not appear in the above expression. This means that the metric information of M is not carried over to M_f . One can see that M_f inherits a complex structure from M . The map \mathcal{S} given by $\mathcal{S}:a_j \rightarrow -\epsilon_{jk}a_k$, where ϵ_{jk} is the antisymmetric tensor with $\epsilon_{12} = 1$, leaves invariant (7) and the equation for a flat connection. Moreover, $\mathcal{S}^2 = -\mathbf{1}_2$. Hence, \mathcal{S} defines an almost complex structure on M_f . This almost complex structure can be used to define the following (1,1) form on M_f

$$\omega(\delta A, \delta B) = \frac{1}{\pi} \int_M (\delta A_z \wedge \overline{\delta B_z} - \delta B_z \wedge \overline{\delta A_z}). \tag{8}$$

Here, we have used complex coordinates ($z = x_1 + ix_2$), and $\delta A = (\delta A_z + \text{c.c.})$. Clearly, ω is manifestly real. Using $d\delta A = d\delta B = 0$, it can also be shown that ω is a closed form. Thus, \mathcal{S} is a complex structure and ω defines a Kähler form. Among many uses of this Kähler form one can, for example, compute the volume of M_f . Recently, we obtained an expression for the Kähler form on M_N and also, we computed the volume of M_N [8]. We will see that when $A = 4\pi N$, the Kähler form on M_f gets mapped to the Kähler form on M_N .

Although a known fact, in order to setup the stage for the main part of this paper we intend to show that the space of flat $U(1)$ connections on M modulo gauge transformations, i.e. M_f , is parametrized by the Picard variety, \tilde{J} , of M [12]. \tilde{J} is dual to the Jacobian J of M . J is a $2g$ -dimensional real torus. Let $\nu_i, (i = 1, \dots, 2g)$ be a basis of $2g$ one-cycles of M . Let $w_\rho, (\rho = 1, \dots, g)$ be a basis of g holomorphic one-forms on M . Define the period matrix $W = (w_{\rho i})$, where $w_{\rho i} = \oint_{\nu_i} w_\rho, (\rho = 1, \dots, g, i = 1, \dots, 2g)$. The columns $\tilde{W}_i, (i = 1, \dots, 2g)$ of W can be thought of as spanning a $2g$ -dimensional lattice in \mathbb{C}^g . J is defined as the torus \mathbb{C}^g/W . Riemann's bilinear relations can be used to normalize the period matrix W [13]. One can choose W such that (W^t, \overline{W}^t) is a symmetric matrix with $\text{Im}(W) > 0$. Further, the elements of W can be restricted to satisfy $w_{\rho i} = \delta_{\rho i}$ for $\rho = 1, \dots, g$ and $i = 1, \dots, g$, the remaining elements being arbitrary with positive imaginary parts [13]. Let us choose a basis of $2g$ one-cocycles α_i such that $\oint_{\nu_j} \alpha_i = \delta_{ij}, (i, j = 1, \dots, 2g)$. These one-cocycles are the $2g$

generators of the cohomology group $H^1(M, \mathbb{Z})$. A canonical basis of ν_i can be chosen such that the cocycles also satisfy $\int_M \alpha_i \alpha_j = \delta_{i, j+g}$, where in the integration wedge product is implied. The g holomorphic one-forms can be expressed in terms of α_i as $w_\rho = \sum_{i=1}^{2g} w_{\rho i} \alpha_i$ for $(\rho = 1, \dots, g)$. Reciprocally, α_i can be expressed in terms of w_ρ as $\alpha_i = \sum_{\rho=1}^g (\gamma_{i\rho} w_\rho + \text{c.c.})$, for $(i = 1, \dots, 2g)$ where the matrix $\Gamma = (\gamma_{\rho i})$ is required to satisfy

$$\Gamma^t W + \overline{\Gamma}^t \overline{W} = \mathbf{1}_{2g}. \tag{9}$$

We note that \tilde{J} is defined as the torus \mathbb{C}^g/Γ . Hence, \tilde{J} is dual to J .

Let $c_\rho, (\rho = 1, \dots, g)$ denote complex coordinates on M_f (that M_f is $2g$ -dimensional will be evident below). Then a real flat connection A_f (modulo gauge transformations) can be expressed as

$$A_f = 2\pi \sum_{\rho=1}^g (c_\rho w_\rho + \text{c.c.}). \tag{10}$$

Imposing Gauss' law on the flat connections one can see that the above is the most general expansion for a flat connection. The holonomy, h_j , of A_f around a one-cycle ν_j is given by

$$h_j = \exp\left(i \oint_{\nu_j} A_f\right) = \exp\left(2\pi i \sum_{\rho=1}^g (c_\rho w_{\rho j} + \text{c.c.})\right). \tag{11}$$

As noted earlier, for a complete specification of A_f one needs to specify all of the holonomies h_j . Eq. (11) implies that h_j is periodic with the period matrix being $\Lambda = (\lambda_{\rho i}), (\rho = 1, \dots, g, i = 1, \dots, 2g)$, say. Then the $2g$ columns of Λ span a $2g$ -dimensional lattice in \mathbb{C}^g . Thus, M_f is parametrized by a $2g$ -dimensional real torus \mathbb{C}^g/Λ . Further, the following relation for Λ is implied by (11)

$$\Lambda^t W + \overline{\Lambda}^t \overline{W} = \mathbf{1}_{2g}. \tag{12}$$

Comparing the above equation with (9) one gets $\Lambda = \Gamma$. This identifies M_f with \tilde{J} .

There are g independent holomorphic one-forms on M_f . Using c_ρ as coordinates of M_f , these are given by $dc_\rho, (\rho = 1, \dots, g)$. A basis of one-cycles in M_f is given by the $2g$ lines $tA_i, 0 \leq t \leq 1$, with A_i identified with 0 to produce a closed loop. Then

from the discussion preceding (9), one deduces that the $2g$ generators of $H^1(M_f, \mathbf{Z})$ are given by $\tilde{\xi}_i = \sum_{\rho=1}^g (w_{\rho i} dc_{\rho} + \text{c.c.})$. Now from (8), we get the following Kähler form on M_f

$$\tilde{\omega} = 4\pi \sum_{\rho, \rho', i=1}^g (w_{\rho i} \bar{w}_{\rho' i+g} - \bar{w}_{\rho' i} w_{\rho i+g}) dc_{\rho} \wedge d\bar{c}_{\rho'}. \tag{13}$$

In terms of $\tilde{\xi}_i$, $\tilde{\omega}$ can be written as

$$\tilde{\omega} = 4\pi \sum_{i=1}^g \tilde{\xi}_i \tilde{\xi}_{i+g}. \tag{14}$$

In obtaining the above from (13) we have used the Riemann bilinear relations. The volume of M_f is then

$$\text{Vol}_f = \frac{1}{g!} \int_{M_f} \tilde{\omega}^g = (4\pi)^g \tag{15}$$

where use has been made of the fact that $\int_{M_f} \prod_{i=1}^g \tilde{\xi}_i \tilde{\xi}_{i+g} = 1$. It is useful to notice that the volumes of J and \tilde{J} are the same. The computation of the volume of the space of flat connections on a compact Riemann surface is not new. For $SU(2)$ and $SO(3)$ Yang-Mills theory, Witten [14] computed the volume of the space of flat connections by a remarkable use of the Verlinde formula [15] in conformal field theory.

3. The general formula for the Kähler form on the vortex moduli space M_N , when ϕ is non-vanishing, is [8]

$$\omega = (A - 4\pi N)\eta + 4\pi \sum_{i=1}^g \xi_i \xi_{i+g} \tag{16}$$

where η is an area form on M_N normalized to unity and $\xi_i, (i = 1, \dots, 2g)$ are the $2g$ generators of $H^1(M_N, \mathbf{Z})$. When $\phi = 0$ is zero, i.e. $A = 4\pi N$, the Kähler form in (16) reduces to

$$\omega = 4\pi \sum_{i=1}^g \xi_i \xi_{i+g}. \tag{17}$$

At this point we should remind the reader that this Kähler form is defined on M_N , not on \tilde{J} whose tangent space coincides with the tangent space of the moduli space of vortices when $\phi = 0$. However, it can be shown that the Kähler form ω on M_N is

mapped in a one-to-one way to the Kähler form $\tilde{\omega}$ on J . As \tilde{J} is isomorphic to J it is enough to establish an isomorphism between $H^1(M_N, \mathbf{Z})$ and $H^1(J, \mathbf{Z})$. First, notice that Jacobi’s inversion theorem [13] implies that there is an isomorphism between J and M_g where M_g is the moduli space of g vortices, and g is the genus of M . This implies the isomorphism between $H^1(J, \mathbf{Z})$ and $H^1(M_g, \mathbf{Z})$. Next, using the Lefschetz hyperplane section theorem [13], one sees that there is an isomorphism between $H^1(M_N, \mathbf{Z})$ and $H^1(J, \mathbf{Z})$ for $N \geq g$ and $g > 1$. The isomorphism for other values of N and g can also be easily established by arguments used in [16].

It is of interest to see if one can relate Vol_f to the volume of M_N near Bradlow’s limit when $\epsilon = A - 4\pi N$ is a small positive quantity. For genus $g \geq 1$ and $N > 2g - 1$, M_N has a bundle structure, where the base is J , and the fibre is CP_{N-g} . For $N \leq g$, M_N is analytically homeomorphic to a $2N$ -dimensional submanifold of the Jacobian. Generically, the volume of M_N is not just a product of the volume of the base and the volume of the fibre. The volume of M_N as computed in [8] is

$$\text{Vol}_N = (A - 4\pi N)^{N-g} \times \sum_{i=0}^g \left(\frac{(4\pi)^i (A - 4\pi N)^{g-i} g!}{(N-i)!(g-i)!i!} \right). \tag{18}$$

In this formula $N \geq g$. It is easy to write an analogous formula for $N < g$. Near Bradlow’s limit the above volume can be written as

$$\text{Vol}_N = (4\pi)^g \frac{\epsilon^{N-g}}{(N-g)!} + O(\epsilon^{N-g+1}). \tag{19}$$

Neglecting the higher order corrections the above can be written as

$$\text{Vol}_N = \text{Vol}_f \times \frac{\epsilon^{N-g}}{(N-g)!} \tag{20}$$

where the factor $\epsilon^{N-g}/(N-g)!$ can be thought of as a contribution coming from the fibre CP_{N-g} . Indeed, using (16) and the cohomology class of the

fibre one can show that the volume of the fibre is $(A - 4\pi N)^{N-g}/(N-g)!$.

In conclusion, we would like to clarify the following apparent puzzle. In computing the Kähler form (16), one needs to extract the non-singular parts of the expressions like $\partial_z \log|\phi|^2$ around the zeros of ϕ . This, however, does not invalidate the derivation of (16) when $\phi = 0$ as one may think. From the Bogomol'nyi equations one can always express $\partial_z \log|\phi|^2$ in terms of the gauge potentials, which can in principle be used to derive (16) regardless of whether ϕ is zero or not.

Acknowledgements

I would like to thank my supervisor Dr N.S. Manton for his continuous guidance and very helpful discussions. Also thanks to P. Irwin for critical comments on this manuscript. This work was supported by the Overseas Research Council, the Cambridge Commonwealth Trust and Wolfson college.

References

- [1] E.B. Bogomol'nyi, Sov. J. Nucl. Phys. 24 (1976) 449.
- [2] C.H. Taubes, Comm. Math. Phys. 72 (1980) 277.
- [3] A. Jaffe, C.H. Taubes, Vortices and Monopoles, Birkhäuser, Boston, 1980.
- [4] N.S. Manton, Nucl. Phys. B 400 (1993) 624.
- [5] N.S. Manton, Phys. Lett. B 110 (1982) 54.
- [6] T.M. Samols, Comm. Math. Phys. 145 (1992) 149; Ph.D Thesis, Cambridge University 1990, unpublished.
- [7] P.A. Shah, N.S. Manton, J. Math. Phys. 35 (1994) 1171.
- [8] N.S. Manton, S.M. Nasir, DAMTP preprint DAMTP-97-90, 1997.
- [9] P.A. Shah, Nucl. Phys. B 438 (1995) 589.
- [10] S. Bradlow, Comm. Math. Phys. 135 (1990) 1.
- [11] Y. Aharonov, D. Bohm, Phys. Rev. 115 (1959) 485.
- [12] R.C. Gunning, Lectures on Riemann Surfaces, Princeton University Press, New Jersey, 1966.
- [13] P. Griffiths, J. Harris, Principles of Algebraic Geometry, John Wiley & Sons, Inc., New York, 1978.
- [14] E. Witten, Comm. Math. Phys. 141 (1991) 153.
- [15] E. Verlinde, Nucl. Phys. B 300 (1988) 360.
- [16] I.G. MacDonald, Topology 1 (1962) 319.