Nonlinearity **12** (1999) 851–865. Printed in the UK PII: S0951-7715(99)96371-5

Conservation laws in a first-order dynamical system of vortices

N S Manton and S M Nasir

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

E-mail: N.S.Manton@damtp.cam.ac.uk and nasir@ml.kva.se

Received 30 July 1998, in final form 19 March 1999
Recommended by S Kida

Abstract. The conservation laws for linear and angular momenta in a Schrödinger–Chern–Simons field theory modelling vortex dynamics in planar superconductors are studied. In analogy with fluid vortices it is possible to express the linear and angular momenta as low moments of vorticity. The conservation laws are shown to be consistent with those obtained in the moduli space approximation for vortex dynamics, valid close to the Bogomol'nyi limit. For Bogomol'nyi vortices, the relevant moments of vorticity can be evaluated fairly explicitly, as can the integral of $\log |\phi|^2$, where ϕ is the scalar field. Conservation of angular momentum prevents a single vortex from escaping to infinity.

PACS numbers: 1127, 1130, 7420D

1. Introduction

Recently, a (2+1)-dimensional field theory of a complex scalar field, with $U(1)$ gauge invariance, was proposed to describe the non-dissipative dynamics of magnetic flux vortices in thin-film superconductors [9]. The Lagrangian is of first-order Schrödinger–Chern–Simons type, containing terms linear in the first time derivatives of the fields, but no quadratic terms. Interestingly, in this theory two vortices orbit around each other. It has been argued that such motion occurs in superconductors at very low temperature [1, 14].

For time-independent fields, the Lagrangian reduces to the standard Ginzburg–Landau energy functional, so for certain values of the coupling constants, there are static multi-vortex solutions obeying Bogomol'nyi equations [4]. Such solutions describe vortices all with the same sign of their quantized magnetic flux. The space of N-vortex solutions, whose parameters are just the vortex positions, is known as the N-vortex moduli space, and is $2N$-dimensional. To understand the vortex motion when the coupling constants have slightly different values, we use the moduli space approach to soliton dynamics, originally applied to Yang–Mills–Higgs monopoles [8]. Here, the vortex dynamics is approximated by a reduced, finite-dimensional dynamical system, obtained by projecting the field dynamics onto the moduli space of Bogomol'nyi vortices. The dynamical variables of the reduced system are just the time-varying vortex positions.

The main conclusion in [9] was the Lagrangian of the reduced system (equation (2.15) below). From this it is straightforward to obtain conserved quantities of the reduced dynamics, which can be interpreted as the linear and angular momenta. The linear momentum turns out to be related to the mean of the vortex positions, which is not so surprising in a first-order dynamical system, and its conservation implies that the vortices circulate around the mean position. The angular momentum is related to the sum of the squared distances of the individual vortices from the mean position, and we shall show that its conservation implies that no vortex can escape to infinity.

The conserved linear and angular momenta of the reduced dynamics were not directly related to the linear and angular momenta of the parent field theory in [9]. This omission is rectified here. The conserved quantities of the field theory have to be derived with care, in order to be gauge invariant. The naive canonical linear and angular momenta are not gauge invariant; moreover, they are not even conserved if the field configuration has non-trivial topology, because of currents at infinity. The relevant conserved quantities have been obtained by Hassaïne *et al* [5] by identifying the field theory with the Jackiw–Pi model [7] in a background field. Here, we obtain the conservation laws more directly using Noether's theorem, and we clarify the issues of gauge invariance and currents at infinity. Following Papanicolaou and Tomaras [10], who studied conservation laws in a similar model, we also express the linear and angular momenta as moments of vorticity. This establishes an analogy between our model of magnetic flux vortices and models of fluid vortices [3]. By evaluating these moments of vorticity for fields satisfying the Bogomol'nyi equations, we rederive the linear and angular momenta of the reduced vortex dynamics.

Evaluating the moments of vorticity is mostly a straightforward application of Green's lemma; however, we encounter one noteworthy result. We obtain an expression for the integral over the plane of $\log|\phi|^2$, where ϕ is the scalar field. This integral converges, despite the logarithmic singularities of the integrand at the vortex locations.

The moduli space approximation has been established rigorously by Stuart [17] in the context of slowly moving Bogomol'nyi vortices in the relativistic Abelian Higgs model. However, in the Schrödinger–Chern–Simons theory of vortices it is plausible, but not yet certain, that the moduli space approximation faithfully describes the vortex dynamics. Obtaining consistent conservation laws provides an important check.

The outline of this paper is as follows. In section 2, we review the field theory, and the reduced Lagrangian of vortex dynamics obtained using the moduli space approximation. In section 3, we obtain the conserved linear and angular momenta in the field theory and express them in terms of vorticity. In section 4, we evaluate these expressions for Bogomol'nyi fields, and compare with the conserved quantities obtained from the reduced Lagrangian. In section 5 we discuss the limit where vortices coalesce into one or more clusters. Section 6 contains our conclusions.

2. The field theory

2.1. The Schrödinger–Chern–Simons Lagrangian

Let ϕ be a complex (Higgs) scalar field representing the condensate of the superconducting electrons and let a_α $(\alpha = 0, 1, 2)$ be the $U(1)$ gauge potential. We will use the subscript 0 to refer to time and the subscripts 1 and 2 to refer to the two directions in space. Sometimes, bold symbols will be used to denote (spatial) two-vectors. The Lagrangian of the model is [9]

$$L = T - V \tag{2.1}$$

where the kinetic energy is

$$T = \int \left(\gamma \frac{\mathrm{i}}{2}(\bar{\phi} D_0\phi - \phi\overline{D_0\phi}) + \mu(Ba_0 + E_2a_1 - E_1a_2) - \gamma a_0 \right) \mathrm{d}^2x \tag{2.2}$$

and the potential energy

$$V = \int \left(\frac{1}{2} B^2 + \frac{1}{2} \overline{D_i \phi} D_i \phi + \frac{\lambda}{8} (1 - |\phi|^2)^2 + a_i J_i^T \right) \mathrm{d}^2 x. \tag{2.3}$$

Here, γ, μ and λ are real constants with λ positive, $D_\alpha \phi = (\partial_\alpha - \mathrm{i} a_\alpha)\phi$ are the components of the covariant derivative of ϕ, $B = \partial_1 a_2 - \partial_2 a_1$ is the magnetic field, $E_i = \partial_i a_0 - \partial_0 a_i$ the electric field, and J_i^T is a constant transport current. We assume the summation convention in the spatial index $i = 1, 2$. The Schrödinger term (with coefficient γ) and the Chern–Simons term (with coefficient μ) define the kinetic energy for the scalar field and gauge potential. The term γa_0, introduced by Barashenkov and Harin [2], allows the possibility of a non-zero condensate in the ground state. The potential energy is the Ginzburg–Landau energy functional. Notice that the kinetic energy contains terms with only the first power of time derivatives. It was shown in [9] that L is Galilean invariant. This implies that given any solution of the field equations in the absence of a transport current, the effect of J_i^T is simply to boost the solution with a velocity $v_i = \frac{1}{\gamma} J_i^T$. Having understood this role of the transport current, we will henceforth suppose it vanishes.

The field equations obtained by varying $\bar{\phi}$, a_i and a_0, respectively, are

$$\mathrm{i}\gamma D_0 \phi = -\frac{1}{2} D_i D_i \phi - \frac{\lambda}{4} \left(1 - |\phi|^2\right) \phi \tag{2.4}$$

$$\epsilon_{ij} \partial_j B = J_i + 2\mu \epsilon_{ij} E_j \tag{2.5}$$

$$2\mu B = \gamma \left(1 - |\phi|^2\right) \tag{2.6}$$

where J_i is the supercurrent defined by

$$J_i = -\frac{\mathrm{i}}{2} \left(\bar{\phi} D_i \phi - \phi \overline{D_i \phi} \right). \tag{2.7}$$

Equation (2.4) is the gauged nonlinear Schrödinger equation, equation (2.5) is the two-dimensional version of Ampère's law and equation (2.6) is a constraint. Such a constraint appears in other Chern–Simons theories [6, 7]. It is useful to note that this constraint is one of the Bogomol'nyi equations for vortices when $\gamma = \mu$ [4]. We shall assume that $1 - |\phi|^2$ and $D_i \phi$ decay rapidly as $|\boldsymbol{x}| \to \infty$. Equations (2.4)–(2.7) imply that $D_0 \phi$, B and E_i decay similarly.

2.2. Vortices

The above field theory admits vortex solutions. Vortices appear whenever there is a non-trivial winding of the map between the boundary circle at spatial infinity and the manifold of ground states of the scalar field, the circle $|\phi| = 1$. The relation between the winding number N and the magnetic flux is

$$\int B \, \mathrm{d}^2 x = 2\pi N. \tag{2.8}$$

N can be interpreted as the vortex number. Henceforth, we suppose $N \geqslant 0$. Later, we define a gauge invariant vorticity $\mathcal{V}$, whose integral is $2\pi N$. However, the vorticity is not simply $\mathcal{V} = B$.

Generally, a solution with N vortices is not static, and we wish to understand how the vortices move. However, it is by now well known that for special values of the couplings a large space of stable, static N-vortex solutions exists, for any $N > 0$ [18]. These solutions satisfy first order Bogomol'nyi equations, as well as the second order Ginzburg–Landau field equations [4]. For the theory here, Bogomol'nyi vortices occur when $\lambda = 1$ and $\gamma = \mu$. The first order Bogomol'nyi equations are

$$(D_1 + \mathrm{i}D_2)\phi = 0 \tag{2.9}$$

$$B = \tfrac{1}{2}(1 - |\phi|^2). \tag{2.10}$$

Solutions of these equations also satisfy (2.4)–(2.7), with $D_0\phi$ and E_i vanishing. Bogomol'nyi vortices do not exert forces on each other, the repulsion of the magnetic fluxes being cancelled by the scalar attraction, and this is why a static configuration of N vortices can exist. The solutions of the Bogomol'nyi equations with winding number N are uniquely specified by the unordered zeros of the scalar field, whose number, counted with multiplicity, is N. These zeros are the vortex positions and we denote them $\{\boldsymbol{x}^s : 1 \leqslant s \leqslant N\}$. The space of solutions, called the N-vortex moduli space, is therefore topologically $\boldsymbol{C}^N/\Sigma_N$, where Σ_N is the permutation group on N objects and the two-dimensional real plane is identified with the complex plane $\boldsymbol{C}$. The N-vortex moduli space is a smooth manifold of dimension $2N$, despite the apparent singularities where vortex positions coalesce.

Hassaïne *et al* have recently discovered stationary Bogomol'nyi-type vortex solutions in this theory with $\gamma \neq \mu$ [5]. The fields satisfy (2.9) and (2.6), and in addition a_0 is proportional to B. One needs $\lambda = 2\gamma/\mu - \gamma^2/\mu^2$, and λ must be positive. These vortices are sources for non-vanishing electric fields. However, we shall not consider these solutions here.

2.3. The reduced Lagrangian for vortex dynamics

We consider the case where λ is close to one and $\gamma = \mu$. We are interested in fields which remain close to N-vortex solutions of the Bogomol'nyi equations, but in which the vortex positions move slowly. In the moduli space approximation to vortex dynamics, one obtains a reduced Lagrangian by simply inserting Bogomol'nyi solutions into (2.1) and taking the vortex positions dependent on time. Let us write ϕ as

$$\phi = \mathrm{e}^{\frac{1}{2}h + \mathrm{i}\chi}. \tag{2.11}$$

h is gauge invariant, and tends to zero at spatial infinity, but is singular at the vortex positions. The first Bogomol'nyi equation (2.9) implies that

$$a_i = \tfrac{1}{2}\epsilon_{ij}\partial_j h + \partial_i \chi. \tag{2.12}$$

From the second Bogomol'nyi equation (2.10), one obtains

$$\partial_i\partial_i h - \mathrm{e}^h + 1 = 4\pi \sum_{s=1}^{N} \delta^2(\boldsymbol{x} - \boldsymbol{x}^s). \tag{2.13}$$

This is the fundamental gauge invariant equation describing Bogomol'nyi vortices. We assume this equation holds, even if the vortex positions $\{\boldsymbol{x}^s\}$ are slowly moving.

Let us now suppose that the vortex positions are distinct, which is the generic case. We shall consider vortex coalescence in section 5. $h = \log|\phi|^2$ has the following expansion around the position of the sth vortex:

$$h = \log|\boldsymbol{x} - \boldsymbol{x}^s|^2 + a^s + \boldsymbol{b}^s \cdot (\boldsymbol{x} - \boldsymbol{x}^s) + \cdots \tag{2.14}$$

where $\{a^s, \boldsymbol{b}^s\}$ are dependent on the positions of the other vortices relative to $\boldsymbol{x}^s$. The usefulness of this expansion was discovered by Samols [12], developing work of Strachan [15]. a^s plays no significant role in what follows, but $\boldsymbol{b}^s$ does. $\boldsymbol{b}^s$ is a measure of the lack of circular symmetry of h around the vortex, and is exponentially small if the other vortices are distant. After various integrations, and suppression of total time derivative terms, one obtains the manifestly gauge invariant reduced Lagrangian [9]

$$L^{\mathrm{red}} = 2\pi\gamma \sum_{s=1}^{N} \left((b_2^s + \tfrac{1}{2}x_2^s)\dot{x}_1^s - (b_1^s + \tfrac{1}{2}x_1^s)\dot{x}_2^s \right) - V^{\mathrm{red}} \tag{2.15}$$

where an overdot denotes time-derivative. This leads directly to equations of motion for the vortex positions. The potential (2.3) simplifies for solutions of the Bogomol'nyi equations to the integral

$$V^{\mathrm{red}} = \frac{\lambda - 1}{8} \int \left(1 - |\phi|^2\right)^2 \, \mathrm{d}^2x \tag{2.16}$$

plus a constant πN, and this is a translationally and rotationally invariant function of the vortex positions. Unfortunately, it appears that V^{red} cannot be simplified to an explicit expression depending only on $\{\boldsymbol{x}^s, a^s, \boldsymbol{b}^s\}$. The functions $\boldsymbol{b}^s$ and V^{red} are not known explicitly as functions of the relative positions of N-vortices, but they can be calculated numerically and this has been performed for two-vortices in [12, 13].

3. Conservation laws in the field theory

The linear and angular momenta for the field theory we are considering here were obtained in [5]. Here, we give an independent derivation from first principles. Let $\{\psi_c\} = \{\phi, \bar{\phi}, a_0, a_1, a_2\}$, where c runs from 1–5. If under a variation of the fields $\delta\psi_c$, the variation of the Lagrangian density, $\mathcal{L}$, is $\delta\mathcal{L} = \partial_\mu \hat{X}^\mu$, then Noether's theorem associates a conserved current with such a variation. (Here and below, we suppress the infinitesimal quantity multiplying such variations.) The Noether current, assuming the summation convention over the index c, is

$$\hat{j}^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu \psi_c)} \delta\psi_c - \hat{X}^\mu. \tag{3.1}$$

By Noether's theorem $\partial_\mu \hat{j}^\mu = 0$, and it follows that the integral of the time component $\hat{j}^0$ is a conserved quantity provided that the spatial components of the current $\hat{j}^1$ and $\hat{j}^2$ fall off sufficiently fast at spatial infinity.

3.1. Energy

The simplest conserved quantity to consider is energy. This is related to invariance under time translation. Naively, the variations of the fields are given by their time derivatives. However, one can supplement this by a gauge transformation with the parameter $-a_0$. The variations of the fields are then

$$\{\delta\psi_c\} = \{D_0\phi, \overline{D_0\phi}, 0, -E_1, -E_2\} \tag{3.2}$$

and the change in $\mathcal{L}$ is $\delta\mathcal{L} = \partial_\mu X^\mu$, where

$$X^0 = \mathcal{L} + \gamma a_0 - \mu a_0 B, \qquad X^1 = -\mu a_0 E_2, \qquad X^2 = \mu a_0 E_1. \tag{3.3}$$

Using (3.1), the energy density is

$$j^0 = \frac{1}{2}B^2 + \frac{1}{2}\overline{D_i\phi} D_i\phi + \frac{\lambda}{8}\left(1 - |\phi|^2\right)^2. \tag{3.4}$$

j^0 is gauge invariant. Moreover, its integral is conserved, because the spatial components of the currents

$$j^1 = -\tfrac{1}{2}\overline{D_1\phi}D_0\phi - \tfrac{1}{2}D_1\phi\overline{D_0\phi} + BE_2 \tag{3.5}$$

$$j^2 = -\tfrac{1}{2}\overline{D_2\phi}D_0\phi - \tfrac{1}{2}D_2\phi\overline{D_0\phi} - BE_1 \tag{3.6}$$

are gauge invariant, and hence decay rapidly at spatial infinity. Thus, the conserved energy is V, as given in (2.3) (recall that the transport current vanishes).

3.2. Momentum

Let us now find the linear momentum P_i, associated with translation in the x_i-direction. First, consider translation in the x_1-direction. The naive variations of the fields are given by the spatial derivatives in the x_1-direction. One supplements this by a gauge transformation with the parameter $-a_1$. The variations of the fields are then

$$\{\delta\psi_c\} = \{D_1\phi, \overline{D_1\phi}, E_1, 0, B\} \tag{3.7}$$

and the change in $\mathcal{L}$ is $\delta\mathcal{L} = \partial_\mu X'^\mu$, where

$$X'^0 = -\mu a_1 B + \gamma a_1, \qquad X'^1 = \mathcal{L} - \mu a_1 E_2, \qquad X'^2 = \mu a_1 E_1. \tag{3.8}$$

The density of the linear momentum in the x_1-direction, calculated using (3.1), is

$$j'^0 = -\gamma(J_1 + a_1). \tag{3.9}$$

Notice that j'^0 is not gauge invariant. Moreover, the spatial components of the currents are

$$j'^1 = -\gamma\frac{\mathrm{i}}{2}\left(\bar\phi D_0\phi - \phi\overline{D_0\phi}\right) - \frac{1}{2}B^2 - \frac{1}{2}|D_1\phi|^2 + \frac{1}{2}|D_2\phi|^2 + \frac{\lambda}{8}\left(1-|\phi|^2\right)^2 + \gamma a_0 \tag{3.10}$$

and

$$j'^2 = -\tfrac{1}{2}(\overline{D_2\phi}D_1\phi + D_2\phi\overline{D_1\phi}). \tag{3.11}$$

j'^1 is not gauge invariant, and hence does not fall off sufficiently fast at infinity for the integral of j'^0 to be conserved. The remedy for both problems is to note that X'^μ is not uniquely defined, but can be altered by adding total derivative terms. One chooses an improved $\tilde X'^\mu$, with $\partial_\mu X'^\mu = \partial_\mu \tilde X'^\mu$, in such a way that the resulting current is gauge invariant. $\tilde X'^\mu$ can be taken as

$$\tilde X'^0 = X'^0 + \gamma\partial_1(x_2a_2) - \gamma\partial_2(x_2a_1) \tag{3.12}$$

$$\tilde X'^1 = X'^1 + \gamma\partial_2(x_2a_0) - \gamma\partial_0(x_2a_2) \tag{3.13}$$

$$\tilde X'^2 = X'^2 + \gamma\partial_0(x_2a_1) - \gamma\partial_1(x_2a_0). \tag{3.14}$$

Using $\tilde X'^\mu$, the improved density of the linear momentum in the x_1-direction is

$$\tilde j'^0 = -\gamma(J_1 + x_2B) \tag{3.15}$$

and the spatial components of the current are

$$\tilde j'^1 = -\gamma\frac{\mathrm{i}}{2}\left(\bar\phi D_0\phi - \phi\overline{D_0\phi}\right) - \frac{1}{2}B^2 - \frac{1}{2}|D_1\phi|^2 + \frac{1}{2}|D_2\phi|^2 + \frac{\lambda}{8}\left(1-|\phi|^2\right)^2 - \gamma x_2E_2 \tag{3.16}$$

$$\tilde j'^2 = -\tfrac{1}{2}(\overline{D_2\phi}D_1\phi + D_2\phi\overline{D_1\phi}) + \gamma x_2E_1. \tag{3.17}$$

Clearly, $\tilde j'^1$ and $\tilde j'^2$ are now gauge invariant and fall off sufficiently fast at spatial infinity. Similarly, one can consider translations in the x_2-direction. From (3.15) and its analogue for x_2-translations, one concludes that the conserved linear momentum is

$$P_i = \gamma\int(J_i + \epsilon_{ij}x_jB)\,\mathrm{d}^2x. \tag{3.18}$$

(The choice of sign, here and in (3.26) below, is deliberate, and ensures agreement between the conservation laws in the field theory and those in the reduced theory. It is made because the field variations are due to passive coordinate variations, whereas later we actively vary the vortex positions in the reduced theory.)

3.3. Angular momentum

Next, let us obtain the conservation law for angular momentum, M, by considering the generator of rotations, $\epsilon_{ij}x_i\partial_j$, combined with a gauge transformation with parameter $-\epsilon_{ij}x_ia_j$. Here, care is needed to evaluate the Lie derivatives correctly on the scalar field and gauge potential. The variations of the fields are

$$\{\delta\psi_c\} = \{\epsilon_{ij}x_iD_j\phi, \epsilon_{ij}x_i\overline{D_j\phi}, \epsilon_{ij}x_iE_j, -x_1B, -x_2B\}. \tag{3.19}$$

The change in $\mathcal{L}$ is $\delta\mathcal{L} = \partial_\mu X''^\mu$, where

$$\begin{aligned} X''^0 &= -\epsilon_{ij}x_ia_j(\mu B - \gamma), \\ X''^1 &= -x_2\mathcal{L} - \mu\epsilon_{ij}x_ia_jE_2, \\ X''^2 &= x_1\mathcal{L} + \mu\epsilon_{ij}x_ia_jE_1. \end{aligned} \tag{3.20}$$

The angular momentum density obtained using (3.1) is

$$j''^0 = -\gamma\epsilon_{ij}x_i(J_j + a_j). \tag{3.21}$$

Neither this density nor the spatial components of the current are gauge invariant, nor do they fall off sufficiently fast at spatial infinity. Again, the remedy is to find an improved $\tilde{X}''^\mu$, with $\partial_\mu X''^\mu = \partial_\mu\tilde{X}''^\mu$. One may take

$$\tilde{X}''^0 = X''^0 - \gamma\partial_1(a_2r^2/2) + \gamma\partial_2(a_1r^2/2) \tag{3.22}$$

$$\tilde{X}''^1 = X''^1 - \gamma\partial_2(a_0r^2/2) + \gamma\partial_0(a_2r^2/2) \tag{3.23}$$

$$\tilde{X}''^2 = X''^2 - \gamma\partial_0(a_1r^2/2) + \gamma\partial_1(a_0r^2/2). \tag{3.24}$$

Then, the improved angular momentum density is

$$\tilde{j}''^0 = -\gamma(\epsilon_{ij}x_iJ_j - \tfrac{1}{2}r^2B) \tag{3.25}$$

which is clearly gauge invariant. Likewise, $\tilde{j}''^1$ and $\tilde{j}''^2$ are gauge invariant and do fall off sufficiently fast. The conserved angular momentum is therefore

$$M = -\int \tilde{j}''^0\,\mathrm{d}^2x = \gamma\int(\epsilon_{ij}x_iJ_j - \tfrac{1}{2}r^2B)\,\mathrm{d}^2x. \tag{3.26}$$

3.4. Vorticity

Let us now define the vorticity

$$\mathcal{V} = \epsilon_{ij}\partial_iJ_j + B. \tag{3.27}$$

Substituting for J_i, using (2.7), the vorticity can be written as

$$\mathcal{V} = -\mathrm{i}\epsilon_{ij}\overline{D_i\phi}D_j\phi + B(1 - |\phi|^2), \tag{3.28}$$

which is the definition in [10], and is a gauge invariant generalization of the notion of vorticity discussed in [11]. In the sector with vortex number N,

$$\int\mathcal{V}\,\mathrm{d}^2x = 2\pi N, \tag{3.29}$$

using (3.27) and Stokes' theorem. Integrating by parts, one may express the linear and angular momenta as the following moments of the vorticity

$$P_i = \gamma \epsilon_{ij} \int x_j \mathcal{V}\, \mathrm{d}^2 x \tag{3.30}$$

and

$$M = -\frac{\gamma}{2} \int r^2 \mathcal{V}\, \mathrm{d}^2 x. \tag{3.31}$$

We noted in the introduction that in a first-order dynamical system the linear momentum can often be taken as a measure of position. Formula (3.30) exemplifies this idea. The components of momentum are proportional to the components of the centre of vorticity. Equations (3.30) and (3.29) imply that $R_i = -\frac{1}{2\pi N\gamma}\epsilon_{ij} P_j$ is the centre of vorticity [10], and it does not move.

The conservation of the angular momentum (3.31) shows that no net vorticity can escape to infinity, assuming that singularities do not form, and therefore a vortex cannot escape to infinity, unless accompanied by an anti-vortex.

The vorticity has the following interesting property for Bogomol'nyi vortices. Substituting (2.10) in (2.3), the energy density for Bogomol'nyi vortices is seen to be

$$\mathcal{E}^{\mathrm{Bog}} = \tfrac{1}{2}|D_i\phi|^2 + B^2. \tag{3.32}$$

On the other hand, using both Bogomol'nyi equations, the vorticity (3.28) can be rewritten as

$$\mathcal{V} = |D_i\phi|^2 + 2B^2. \tag{3.33}$$

Thus, for Bogomol'nyi vortices $\mathcal{V} = 2\mathcal{E}^{\mathrm{Bog}}$.

4. Conservation laws in the reduced dynamics

4.1. The conserved quantities

Conservation laws of the reduced dynamics can be obtained directly from L^{red}, equation (2.15). (Note that the discussion of conservation laws at the end of [9] is slightly wrong.) In general, a variation $\delta x_i^s = \xi_i^s$ is a symmetry if $\delta L^{\mathrm{red}} = \frac{\mathrm{d}}{\mathrm{d}t} X$ for some X. Noether's theorem states that

$$\sum_{s=1}^{N} \frac{\partial L^{\mathrm{red}}}{\partial \dot{x}_i^s} \xi_i^s - X \tag{4.1}$$

is then conserved.

A translation of all the vortex positions in the x_1-direction is a symmetry. Here, for all s,

$$\delta x_1^s = 1, \qquad \delta x_2^s = 0, \tag{4.2}$$

and $\boldsymbol{b}^s$ and V^{red} are invariant. One finds that $X = -\pi\gamma \sum_{s=1}^N x_2^s$, and the conserved component of momentum is

$$P_1^{\mathrm{red}} = 2\pi\gamma \sum_{s=1}^{N} (b_2^s + x_2^s). \tag{4.3}$$

Similarly, translation in the x_2-direction is a symmetry, and

$$P_2^{\mathrm{red}} = -2\pi\gamma \sum_{s=1}^{N} (b_1^s + x_1^s) \tag{4.4}$$

is conserved. It was shown by Samols that $\sum_{s=1}^{N} \boldsymbol{b}^s = 0$. The linear momentum in the reduced dynamics is therefore simply

$$P_i^{\text{red}} = 2\pi\gamma\epsilon_{ij} \sum_{s=1}^{N} x_j^s, \tag{4.5}$$

and directly related to the mean of the vortex positions. Conservation of momentum implies that the vortices circulate about their fixed mean position.

There is also symmetry under a rotation, where, for all s

$$\delta x_1^s = -x_2^s, \qquad \delta x_2^s = x_1^s. \tag{4.6}$$

V^{red} is invariant, but the rotation leads to the variations

$$\delta b_1^s = -b_2^s, \qquad \delta b_2^s = b_1^s. \tag{4.7}$$

It follows that L^{red} is strictly invariant, with $X = 0$, so one has the following conserved angular momentum in the reduced dynamics

$$M^{\text{red}} = -2\pi\gamma \sum_{s=1}^{N} \left(\tfrac{1}{2}\boldsymbol{x}^s \cdot \boldsymbol{x}^s + \boldsymbol{b}^s \cdot \boldsymbol{x}^s\right). \tag{4.8}$$

This conservation law implies that no vortex can escape to infinity, as is shown in section 5.

4.2. Comparison with the field theory

Here, we compare the conserved quantities in the field theory with the corresponding conserved quantities obtained directly from the reduced Lagrangian, L^{red}. We assume that the fields satisfy the Bogomol'nyi equations at all times, possibly with time-varying vortex positions, and evaluate the conserved quantities for such fields. This is sensible if λ is close to 1 and $\mu = \gamma$.

First of all, for fields satisfying the Bogomol'nyi equations, the conserved field energy is $E = \pi N + V^{\text{red}}$. This is consistent with the reduced dynamics, where the Hamiltonian is simply V^{red} (the constant πN is dropped), and V^{red} is conserved.

The main task is to evaluate the moments of vorticity (3.30) and (3.31), defining the linear and angular momentum. Using (2.11) and (2.12), it can be shown that for solutions of the Bogomol'nyi equations

$$J_i = -\tfrac{1}{2}\epsilon_{ij}\partial_j h \mathrm{e}^h, \qquad B = -\tfrac{1}{2}\partial_i\partial_i h. \tag{4.9}$$

The vorticity $\mathcal{V}$, defined in (3.27), becomes

$$\mathcal{V} = \tfrac{1}{2}\partial_i\partial_i(\mathrm{e}^h - h) = \tfrac{1}{2}\partial_i\partial_i(\mathrm{e}^h - h - 1), \tag{4.10}$$

where the second expression is more useful, as the quantity in brackets vanishes at infinity. Another expression for $\mathcal{V}$ is

$$\mathcal{V} = \tfrac{1}{2}\partial_i(\partial_i h(\mathrm{e}^h - 1)) = \tfrac{1}{2}\partial_i(\partial_i h \partial_j\partial_j h), \tag{4.11}$$

where use has been made of (2.13) and temporarily we ignore the logarithmic singularities of h. Note that $\mathcal{V}$ is a smooth function despite the singularities of h.

In what follows we still suppose that ϕ has N simple zeros. In order to carry out the integrals involving moments of $\mathcal{V}$ let us remove, from $\boldsymbol{R}^2$, N discs of small radius ϵ centred at the vortex positions, and call the resulting region $\boldsymbol{R}_0^2$. As $\mathcal{V}$ is a smooth function, integrations over the discs will be $\mathrm{O}(\epsilon^2)$ or smaller, and hence can be neglected in the limit $\epsilon \to 0$. Thus, in the following, the effective region of integration is $\boldsymbol{R}_0^2$, and the singularities of h may be ignored in the formulae (4.10) and (4.11) for $\mathcal{V}$.

Let C^s, where s runs from 1 to N, denote the boundary of the disc around the sth vortex position $\boldsymbol{x}^s$ and let C^0 denote the boundary circle at spatial infinity. Further, let θ^s be the polar angle relative to $\boldsymbol{x}^s$ with $\theta^s = 0$ in the positive x_1-direction. Then, the outward unit normal along C^s is $\boldsymbol{n}^s = (\cos\theta^s, \sin\theta^s)$ and the coordinates of points on C^s can be written as $x_i = x_i^s + \epsilon n_i^s$. The differential line element on C^s is $\mathrm{d}l = \epsilon\, \mathrm{d}\theta^s$.

Now, using (4.10), and remembering the discussion above, the linear momentum (3.30) can be written with $\mathrm{O}(\epsilon^2)$ error as

$$P_i = \frac{\gamma}{2}\epsilon_{ij}\int_{\boldsymbol{R}_0^2}\left(x_j\partial_k\partial_k(\mathrm{e}^h - h - 1)\right)\mathrm{d}^2x. \tag{4.12}$$

Using Green's lemma in two dimensions,

$$P_i = -\frac{\gamma}{2}\epsilon_{ij}\sum_{s=1}^{N}\int_{C^s}\left(x_j\partial_k(\mathrm{e}^h - h - 1) - (\mathrm{e}^h - h - 1)\partial_k x_j\right)n_k^s\,\mathrm{d}l \tag{4.13}$$

$$= -\frac{\gamma}{2}\epsilon_{ij}\sum_{s=1}^{N}\int_{C^s}\left(x_j\partial_k h(\mathrm{e}^h - 1)n_k^s - (\mathrm{e}^h - h - 1)n_j^s\right)\mathrm{d}l. \tag{4.14}$$

There is no contribution from C^0, the circle at infinity, as $\mathrm{e}^h - h - 1$ vanishes there. In calculating the integrals along C^s we ignore terms which are $\mathrm{O}(\epsilon)$ or smaller. On C^s, one finds from (2.14) that

$$\partial_k h = \frac{2n_k^s}{\epsilon} + b_k^s + \cdots, \tag{4.15}$$

and $\mathrm{e}^h = \mathrm{O}(\epsilon^2)$. Therefore

$$P_i = -\frac{\gamma}{2}\epsilon_{ij}\sum_{s=1}^{N}\int_{C^s}\left(-(x_j^s + \epsilon n_j^s)\left(\frac{2n_k^s}{\epsilon} + b_k^s\right)n_k^s + (\log\epsilon^2 + a_s + 1)n_j^s\right)\mathrm{d}l. \tag{4.16}$$

Noting that $\int_{C^s} n_j^s\,\mathrm{d}l = 0$ and $\int_{C^s} n_j^s n_k^s\,\mathrm{d}l = \pi\epsilon\delta_{jk}$, we conclude that

$$P_i = 2\pi\gamma\epsilon_{ij}\sum_{s=1}^{N}x_j^s. \tag{4.17}$$

This reproduces the expression (4.5) for the linear momentum, derived in the reduced dynamics.

We turn now to the angular momentum M. Note that if we used expression (4.10) for the vorticity $\mathcal{V}$, and applied Green's lemma to integrate $r^2\mathcal{V}$, we would require the integral over the plane of h. This integral is discussed below, but for the moment we use the alternative expression (4.11) for $\mathcal{V}$. Then we have the following useful identity

$$r^2\mathcal{V} = \tfrac{1}{2}r^2\partial_i(\partial_i h\partial_j\partial_j h) = \tfrac{1}{2}\partial_i q_i, \tag{4.18}$$

where

$$q_i = r^2\partial_i h\partial_j\partial_j h - 2x_j\partial_j h\partial_i h + x_i\partial_j h\partial_j h. \tag{4.19}$$

We now use the divergence theorem. As $r^2\mathcal{V}$ is a smooth function, we follow the same procedure as in evaluating P_i, namely, remove N small discs centred at the positions of the vortices. With $\mathrm{O}(\epsilon^2)$ error,

$$M = -\frac{\gamma}{2}\int r^2\mathcal{V}\,\mathrm{d}^2x = -\frac{\gamma}{4}\int_{\boldsymbol{R}_0^2}\partial_i q_i\,\mathrm{d}^2x = \frac{\gamma}{4}\sum_{s=1}^{N}\int_{C^s}q_i n_i^s\,\mathrm{d}l. \tag{4.20}$$

Again there is no contribution coming from C^0, the circle at infinity, as $\partial_i h$ vanishes there. We decompose the last sum as

$$M = \frac{\gamma}{4}\sum_{s=1}^{N}(I_1^s - 2I_2^s + I_3^s), \tag{4.21}$$

where

$$I_1^s = \int_{C^s} (r^2 \partial_i h \partial_j \partial_j h) n_i^s \, \mathrm{d}l, \tag{4.22}$$

$$I_2^s = \int_{C^s} (x_j \partial_j h \partial_i h) n_i^s \, \mathrm{d}l, \tag{4.23}$$

and

$$I_3^s = \int_{C^s} (x_i \partial_j h \partial_j h) n_i^s \, \mathrm{d}l. \tag{4.24}$$

Noting from (2.14) that on C^s, $\partial_i \partial_i h = -1 + \mathrm{O}(\epsilon^2)$, we obtain

$$I_1^s = \int_{C^s} (x_k^s x_k^s + 2\epsilon n_k^s x_k^s) \left(\frac{2}{\epsilon} + b_i^s n_i^s \right) (-1) \, \mathrm{d}l = -4\pi x_k^s x_k^s \tag{4.25}$$

where, as usual, terms of $\mathrm{O}(\epsilon)$ or smaller have been neglected. Similarly,

$$I_2^s = \int_{C^s} \left(\frac{2}{\epsilon} x_j^s n_j^s + 2 + x_j^s b_j^s \right) \left(\frac{2}{\epsilon} + b_i^s n_i^s \right) \mathrm{d}l = 8\pi + 6\pi b_j^s x_j^s, \tag{4.26}$$

and

$$I_3^s = \int_{C^s} (x_i^s n_i^s + \epsilon) \left(\frac{4}{\epsilon^2} + \frac{4}{\epsilon} b_j^s n_j^s + b_j^s b_j^s \right) \mathrm{d}l = 8\pi + 4\pi b_j^s x_j^s. \tag{4.27}$$

Thus, putting all the above integrals together,

$$M = -2\pi\gamma \sum_{s=1}^{N} (\tfrac{1}{2} \boldsymbol{x}^s \cdot \boldsymbol{x}^s + \boldsymbol{b}^s \cdot \boldsymbol{x}^s + 1), \tag{4.28}$$

which apart from a constant additive term agrees with (4.8).

The constant term which appears in (4.28) is an important contribution, and is a consequence of the careful treatment of the field theory. (Its absence in the reduced dynamics results from discarding various terms that do not affect the equations of motion.) It implies that a single vortex situated at the origin has total angular momentum $-2\pi\gamma$. Thus, the vortex has an intrinsic spin.

4.3. *The integral of h*

It is rather remarkable that for Bogomol'nyi vortices, where h satisfies (2.13) and the boundary condition $h \to 0$ as $|\boldsymbol{x}| \to \infty$, it is possible to evaluate the integral of h over the plane. The integral is finite, despite the logarithmic singularities of h. The result can be derived from the formula (4.28) for M, by recalculating the integral of $r^2 \mathcal{V}$ using the expression (4.10) for $\mathcal{V}$. However, there is a simpler method that we present here.

The result follows from the equation, valid away from the singularities of h,

$$2h = \partial_i \{ x_j \partial_j h \partial_i h - \tfrac{1}{2} x_i \partial_j h \partial_j h + x_i (1 - \mathrm{e}^h + h) \} - 2(1 - \mathrm{e}^h). \tag{4.29}$$

This is easily verified, after noting that the first two terms on the right-hand side give $x_j \partial_j h \partial_i \partial_i h$, which may be replaced by $-x_j \partial_j h (1 - \mathrm{e}^h)$. From (2.13) and (4.9), $1 - \mathrm{e}^h = 2B$, so

$$\int (1 - \mathrm{e}^h) \, \mathrm{d}^2 x = 4\pi N. \tag{4.30}$$

The integral of the rest of the right-hand side of (4.29) is evaluated with the help of the divergence theorem. There is no contribution from the circle at infinity. The contributions of the first two terms from the circles C^s have already appeared in (4.23) and (4.24), and their values are given by (4.26) and (4.27). There is no contribution from the third term, because its $O(\log \epsilon)$ behaviour on C^s is not singular enough. Hence

$$\int h \, d^2x = -2\pi \sum_{s=1}^{N} (\boldsymbol{b}^s \cdot \boldsymbol{x}^s + 3). \tag{4.31}$$

In particular, for one vortex $\int h \, d^2x = -6\pi$.

4.4. Contribution of the supercurrent to the momenta

It is of some interest to separate the contributions of the supercurrent and the magnetic field to the linear and angular momenta, for fields satisfying the Bogomol'nyi equations. Recall that the vorticity is

$$\mathcal{V} = \mathcal{V}_J + B, \tag{4.32}$$

where the contribution due to the supercurrent is

$$\mathcal{V}_J = \epsilon_{ij} \partial_i J_j. \tag{4.33}$$

The integral of $\mathcal{V}_J$ over the plane is zero, so the total vorticity, or vortex number, is entirely due to the magnetic field. From (4.10),

$$\mathcal{V}_J = \tfrac{1}{2} \partial_i \partial_i (e^h) = \tfrac{1}{2} \partial_i \partial_i (\partial_j \partial_j h). \tag{4.34}$$

It is not difficult to show, using similar methods as previously, that for $k = 1, 2$

$$\int x_k \mathcal{V}_J \, d^2x = 0. \tag{4.35}$$

The supercurrent therefore makes no contribution to the linear momentum, and

$$P_i = \gamma \epsilon_{ij} \int x_j B \, d^2x. \tag{4.36}$$

Further, it can be shown that

$$\int (x_k)^2 \mathcal{V}_J \, d^2x = -4\pi N. \tag{4.37}$$

Hence, the supercurrent contribution to the angular momentum M is

$$M_J = -\frac{\gamma}{2} \int r^2 \mathcal{V}_J \, d^2x = 4\pi \gamma N, \tag{4.38}$$

just a constant. The contribution due to B is therefore

$$M_B = -\frac{\gamma}{2} \int r^2 B \, d^2x = -2\pi \gamma \sum_{s=1}^{N} \left(\tfrac{1}{2} \boldsymbol{x}^s \cdot \boldsymbol{x}^s + \boldsymbol{b}^s \cdot \boldsymbol{x}^s + 3 \right), \tag{4.39}$$

and this carries the non-trivial dynamical information.

It is possible to compute the third and the fourth moments of $\mathcal{V}_J$, for fields satisfying the Bogomol'nyi equations. One can show, using (4.34) and Green's lemma, that

$$\int (x_k)^3 \mathcal{V}_J \, d^2x = -12\pi \sum_{s=1}^{N} x_k^s. \tag{4.40}$$

Similarly,

$$\int (x_k)^4 \mathcal{V}_J \, d^2x = 6 \int (x_k)^2 \partial_j \partial_j h \, d^2x = 6 \int \left(\partial_j ((x_k)^2 \partial_j h) - 2\partial_k (x_k h) + 2h \right) d^2x, \tag{4.41}$$

where there is no summation over the index k. Integrating again, and using (4.31), one finds

$$\int (x_k)^4 \mathcal{V}_J \, d^2x = -24\pi \sum_{s=1}^{N} \left((x_k^s)^2 + \boldsymbol{b}^s \cdot \boldsymbol{x}^s + 3 \right). \tag{4.42}$$

5. Coincident vortices

The formula (4.28) for the angular momentum of N vortices, in the reduced dynamics, assumes that no pair of vortices coincide. The same applies to some related formulae, for example the expression (4.31) for the integral of h.

With care, it is possible to take the limit as a cluster of vortices, or possibly all N vortices, become coincident. It is simplest to discuss what happens when all N vortices coincide. Suppose that the positions $\{\boldsymbol{x}^s\}$ are all close together but still distinct, that is $0 < |\boldsymbol{x}^s - \boldsymbol{x}^r| \ll 1$ for each pair $r \neq s$. In the neighbourhood of these positions, the solution of equation (2.13) for h is approximately

$$h = \sum_{s=1}^{N} \log|\boldsymbol{x} - \boldsymbol{x}^s|^2 + a + \boldsymbol{b}\cdot\boldsymbol{x}, \tag{5.1}$$

for some constants a and $\boldsymbol{b}$. This is an exact solution if the terms $1 - \mathrm{e}^h$ in (2.13) are ignored. The term 1 produces a quadratic correction, and e^h is a high-order polynomial producing smaller corrections. The expansion of h about $\boldsymbol{x}^s$ is

$$h = \log|\boldsymbol{x} - \boldsymbol{x}^s|^2 + \sum_{r\neq s} \log|\boldsymbol{x}^s - \boldsymbol{x}^r|^2 + a + \boldsymbol{b}\cdot\boldsymbol{x}^s$$
$$+ \sum_{r\neq s} \frac{2(\boldsymbol{x}^s - \boldsymbol{x}^r)\cdot(\boldsymbol{x} - \boldsymbol{x}^s)}{|\boldsymbol{x}^s - \boldsymbol{x}^r|^2} + \boldsymbol{b}\cdot(\boldsymbol{x} - \boldsymbol{x}^s) + \mathrm{O}(|\boldsymbol{x} - \boldsymbol{x}^s|^2). \tag{5.2}$$

Therefore

$$\boldsymbol{b}^s = \sum_{r\neq s} \frac{2(\boldsymbol{x}^s - \boldsymbol{x}^r)}{|\boldsymbol{x}^s - \boldsymbol{x}^r|^2} + \boldsymbol{b} \tag{5.3}$$

with corrections that tend to zero as the vortices coincide. Note that $\boldsymbol{b}^s$ is singular due to the coalescence of the vortices. Now recall the constraint $\sum_{s=1}^{N} \boldsymbol{b}^s = 0$. The singular terms cancel in pairs in this sum, so $\boldsymbol{b}$ must vanish.

We can now calculate the angular momentum using expression (5.3) for $\boldsymbol{b}^s$, with $\boldsymbol{b} = \boldsymbol{0}$. We obtain first

$$\sum_{s=1}^{N} \boldsymbol{b}^s\cdot\boldsymbol{x}^s = \sum_{s=1}^{N}\sum_{r\neq s} \frac{2(\boldsymbol{x}^s - \boldsymbol{x}^r)\cdot\boldsymbol{x}^s}{|\boldsymbol{x}^s - \boldsymbol{x}^r|^2} = \sum_{1\leqslant r<s\leqslant N} \frac{2(\boldsymbol{x}^s - \boldsymbol{x}^r)\cdot(\boldsymbol{x}^s - \boldsymbol{x}^r)}{|\boldsymbol{x}^s - \boldsymbol{x}^r|^2} = N(N-1). \tag{5.4}$$

Therefore, as all N vortices coincide at the origin, the expression (4.28) for the angular momentum has the finite limit

$$M = -2\pi\gamma N^2, \tag{5.5}$$

which can be interpreted as the spin of the N coincident vortices. In other Chern–Simons theories, it has also been found that N coincident vortices have spin proportional to N^2 [6].

In the same coincident limit,

$$\int h\,\mathrm{d}^2x = -2\pi N(N+2). \tag{5.6}$$

This result follows from (5.4) and (4.31). Alternatively, it can be obtained directly, by using the circularly symmetric form of the equation for h, multiplying by $r^2\frac{\mathrm{d}h}{\mathrm{d}r}$ and integrating, and using the boundary condition that $h \sim 2N\log r$ as $r \to 0$.

One may generalize the preceeding discussion to the situation where one or more smaller clusters of vortices form. Singular terms develop in $\boldsymbol{b}^s$, if other vortices coalesce with the sth. However, $\sum \boldsymbol{b}^s$ and $\sum \boldsymbol{b}^s\cdot\boldsymbol{x}^s$ have finite limits.

The finiteness of the angular momentum as vortices coalesce implies that angular momentum conservation prevents any vortex escaping to infinity. The first term in the expression (4.28) for M would diverge if a vortex escaped, but we have now shown that the second term involving $\sum \boldsymbol{b}^s \cdot \boldsymbol{x}^s$ cannot have a compensating divergence even if a vortex cluster forms. This result is consistent with the more general argument, given earlier, that no net vorticity can escape to infinity in the field theory.

6. Conclusions

In this paper, we have succeeded in obtaining, from first principles, the conservation laws of linear and angular momenta in the Schrödinger–Chern–Simons theory of vortex dynamics proposed in [9]. As for fluid vortices, the linear and angular momenta can be expressed as low moments of a suitably defined vorticity. Our expressions agree with those in [5] in the absence of any transport current. The conserved quantities in the presence of a transport current are those that follow using the Galilean invariance of the dynamics.

For a range of values of the couplings, vortex dynamics in the theory reduces, approximately, to motion in the moduli space of Bogomol'nyi vortices. The expressions for the linear and angular momenta in the reduced dynamics have been shown to agree with those obtained by evaluating the linear and angular momenta in the parent field theory, assuming the fields satisfy the Bogomol'nyi equations. This agreement was not inevitable, and supports the use of the moduli space approximation. Various integrals involving the vorticity have been evaluated explicitly to make the comparison possible. A novel expression for the integral of $h = \log |\phi|^2$ has been found. One consequence of the calculations is that each vortex has a non-zero net spin. As N vortices coalesce, the total spin is N^2 times the spin of one vortex. Our results can probably be extended to a larger range of couplings, by exploiting the electrically excited vortex solutions of Hassaïne *et al* [5].

The conservation of linear momentum implies that the centre of vorticity, which becomes the mean of the vortex positions in the reduced dynamics, does not move. Conservation of angular momentum implies that net vorticity cannot escape to infinity; in particular, no vortex can escape to infinity in the reduced dynamics. Recently, there have been numerical studies of vortices in the theory considered here [16]. The conservation laws should be a useful guide to the accuracy of such numerical studies.

Acknowledgments

NSM is grateful to N Papanicolaou and P Horváthy, and their collaborators, for very helpful discussions and hospitality. He also thanks K Lee for helpful comments.

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