Nonlinearity **11** (1998) 445–466. Printed in the UK PII: S0951-7715(98)86320-2

Study of Bogomol'nyi vortices on a disk

Sazzad Mahmud Nasir†

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

Received 28 July 1997, in final form 3 December 1997
Recommended by H Levine

Abstract. Bogomol'nyi vortices on a disk with Dirichlet boundary conditions are investigated and solutions are shown to exist. The dimension of the moduli space of solutions is twice the vortex number. For one vortex near the boundary of the disk, the moduli space looks like a hyperbolic plane. To check these findings various numerical tests are made on square domains with/without holes.

PACS number: 1110

1. Introduction

Bogomol'nyi vortices arise in two dimensions as topologically stable, static, and finite energy solutions to the critically coupled Abelian Higgs model [3]. It is of interest to know more about their properties and dynamics. The moduli space of Bogomol'nyi vortices, which is a finite-dimensional manifold that parametrizes the solutions, plays an important role in the study of dynamics. According to Manton [10], at low energy—when most of the degrees of freedom remain unexcited—the dynamics of these vortices can be effectively modelled as geodesic motion on the moduli space. This moduli space approximation thus enables us to describe various physical phenomena of Bogomol'nyi vortices [11–13].

Bogomol'nyi vortices have usually been studied either on $\mathbb{R}^2$ or on a compact two-dimensional region without a boundary [14, 15, 4]. However, one may also study vortices on a compact two-dimensional region with a boundary. This may be applicable to the study of superconductivity. In a region with a boundary one is first confronted with the choice of boundary conditions. These choices must be made in such a way as to ensure the existence of vortices. Moreover, it is desirable in the study of vortices in a compact region with a boundary to retain as much as possible the features of vortices on $\mathbb{R}^2$ or in a compact region without a boundary. It is known that for vortices in a compact region without a boundary or in $\mathbb{R}^2$, the energy, as well as the total magnetic flux, is quantized. Thus, for vortices in a compact region with a boundary one would like to see quantization of energy and total magnetic flux.

One can think of two types of natural boundary conditions: Dirichlet boundary conditions or Neumann boundary conditions. For Dirichlet boundary conditions, the Higgs field takes its vacuum expectation value on the boundary. This choice essentially gives the winding necessary for the topological existence of vortices. Besides, it also guarantees

† E-mail: S.M.Nasir@damtp.cam.ac.uk

the quantization of energy, although the total magnetic flux is not quantized. On the other hand, for Neumann boundary conditions, the normal component of the covariant derivative of the Higgs field, vanishes at the boundary. Such a choice may not produce any vortex-like solution at all. In this paper we will consider the Dirichlet case only. As a model of the compact region with a boundary, we will consider a two-dimensional region, M, which is topologically a disk. Occasionally, we will also consider a flat disk—which will be denoted by M as well—with a circular boundary of radius R.

After choosing the boundary conditions, one can then ask questions about the existence of solutions and, if solutions do exist, about the dimension of the moduli space and about what it looks like. It is known that on $\mathbb{R}^2$ [14, 15] or on a compact region without boundary [4, 8] Bogomol'nyi vortices do exist. The dimension of the moduli space is determined by the vortex number, which is the number of zeros of the Higgs field counted by multiplicity. In the sector with vortex number N, this dimension is $2N$. For the Bogomol'nyi vortices on a disk, we show, following Taubes [14, 9], that solutions do exist and the moduli space is $2N$ dimensional. Below, we give numerical evidence supporting this. Incidentally, we should point out that a naive application of index theory in this case gives a different count for the dimension. However, for vortices in $\mathbb{R}^2$ or in a compact region without a boundary, index theory gives the correct dimension.

The choice of Dirichlet boundary conditions makes it possible to explore the moduli space near the boundary. We do this explicitly for one vortex. In this case, near the boundary, one can approximate the nonlinear Bogomol'nyi equation for the Higgs field by a Laplace equation with a source. The well known image method for solving a Laplace equation can be used to obtain the approximate behaviour of the Higgs field near the boundary. The solutions in turn can be used to extract information about the volume, and about the metric of the moduli space near the boundary. The moduli space for one vortex near the boundary is found to be a hyperbolic plane and hence, the volume of the moduli space diverges. This came as a surprise. It might have been expected that for Bogomol'nyi vortices on a finite region, the moduli space should also be a finite region.

This paper is organized as follows. We begin section 2 with a brief introduction to the Abelian Higgs model and the associated Bogomol'nyi equations with the various choices of boundary conditions and their relation to energy and magnetic flux quantization. In section 3 we show how Taubes' work [14, 9] can be adapted to our case to prove the existence of the solutions. This also shows that the dimension of the moduli space is $2N$. In section 4 we compute the volume, and the metric of the moduli space near the boundary. Finally, section 5 describes various numerical tests of the analytic calculations.

2. Bogomol'nyi vortices

We start with a complex scalar field, ϕ, coupled to a $U(1)$ gauge potential, A_μ, in 2+1 dimensions with the metric signature $(1, -1, -1)$. The metric of the spacetime $\mathbb{R} \times M$, can be taken to be of the form

$$\mathrm{d}s^2 = \mathrm{d}x_0^2 - \Omega(x_1, x_2)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2). \tag{2.1}$$

For a flat disk we can set $\Omega = 1$. The Lagrangian density for the Abelian Higgs model is

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D_\mu\phi\overline{D^\mu\phi} - \tfrac{1}{8}\lambda(|\phi|^2 - 1)^2 \tag{2.2}$$

where $D_\mu = \partial_\mu - \mathrm{i}A_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $(\mu, \nu = 0, 1, 2)$ and the units are chosen such that both the gauge-field coupling constant and the mass of the Higgs field are one. The free parameter λ determines the nature of interaction between the vortices. For $\lambda < 1$ the vortices attract—a case relevant to type I superconductivity—and for $\lambda > 1$ they repel—a case relevant to type II superconductivity. For $\lambda = 1$, dubbed the critically coupled case and the case we are considering here, the total force is zero, permitting static multivortices to exist. In this case it is possible to factorize the Lagrangian in such a way that vortices satisfy the first-order Bogomol'nyi equations.

Working in the gauge $A_0 = 0$, the Lagrangian is $L = T - V$ where,

$$T = \tfrac{1}{2} \int_M \mathrm{d}^2x \, (\dot{A}_i \dot{A}_i + \Omega \dot{\phi} \dot{\bar{\phi}}), \qquad (i = 1, 2) \tag{2.3}$$

$$V = \tfrac{1}{2} \int_M \mathrm{d}^2x \, \Omega \, (\tfrac{1}{2} F_{ij} F^{ij} + D_i \phi \overline{D^i \phi} + \tfrac{1}{4}(|\phi|^2 - 1)^2) \tag{2.4}$$

are, respectively, the kinetic and the potential energies. Further, we need to impose Gauss' law. This arises from the equation of motion of A_0 as the following constraint

$$\partial_i \dot{A}_i - \Omega \mathrm{Im}\,(\dot{\phi} \bar{\phi}) = 0, \qquad (i = 1, 2). \tag{2.5}$$

In the static case the total energy, $E = V$, becomes

$$E = \frac{1}{2} \int_M \mathrm{d}^2x \left[(D_1 \pm \mathrm{i} D_2)\phi \, \overline{(D_1 \pm D_2)\phi} + \Omega^{-1} \left\{ F_{12} \pm \frac{\Omega}{2}(|\phi|^2 - 1) \right\}^2 \right.$$
$$\left. \pm \mathrm{i} \{ \partial_2 (\bar{\phi} D_1 \phi) - \partial_1 (\bar{\phi} D_2 \phi) \} \pm F_{12} \right]. \tag{2.6}$$

Bogomol'nyi vortices, which minimize the above energy integral, satisfy the first-order Bogomol'nyi equations:

$$(D_1 \pm \mathrm{i} D_2)\phi = 0 \tag{2.7}$$

$$F_{12} \pm \frac{\Omega}{2}(|\phi|^2 - 1) = 0. \tag{2.8}$$

Here, the positive sign gives rise to vortices and the negative sign to antivortices. One can obtain antivortices from vortices by changing the orientation of the plane. In this paper we will only consider vortices.

Before discussing the choice of the boundary condition, we rewrite the Bogomol'nyi equations in terms of a new field h, where $h = 2 \log |\phi|$. The field h is singular at the zeros of ϕ. From (2.7), expressing the gauge potentials in terms of h and the phase of the Higgs field, and substituting them back into (2.8), we obtain the following equation satisfied by h

$$\triangle h + \Omega (\mathrm{e}^h - 1) = 4\pi \sum_{i=1}^{N} \delta^2(\boldsymbol{x} - \boldsymbol{x}_i) \tag{2.9}$$

where $\boldsymbol{x}_i$ denote the position of the zero of the Higgs field associated with the ith vortex and $\triangle$ is the flat-space Laplacian $\partial_1^2 + \partial_2^2$. This equation will play an important role in proving the existence of the solutions.

The vacuum solution with $E = 0$, corresponds to taking $\phi = 1$ everywhere with the gauge potentials zero. In order to obtain finite-energy Bogomol'nyi vortices in $\mathbb{R}^2$, one usually sets on the boundary, $|\phi| = 1$ and $D_t \phi = 0$, where D_t is the tangential component of the covariant derivative. The first condition makes it possible to classify the vortices into topologically stable sectors determined by vortex number N. The second condition allows us to express the energy solely in terms of the total magnetic flux, Φ. One finds that the energy $E = \pi N$ and the total magnetic flux $\Phi = 2\pi N$, respectively.

As in the $\mathbb{R}^2$ case, for the Bogomol'nyi vortices on a disk one can take Dirichlet conditions

$$|\phi| = 1 \tag{2.10}$$

on the boundary. As in the $\mathbb{R}^2$ case, this gives the necessary winding number for vortices to be classified into different topological sectors. However, unlike the $\mathbb{R}^2$ case, one cannot impose $D_t\phi = 0$ on the boundary. This can be seen heuristically as follows. If $D_t\phi = 0$, the Bogomol'nyi equation (2.7) would imply that $D_n\phi$, which is the normal component of the covariant derivative of the Higgs field, vanishes on the boundary as well. This means that $\partial_n|\phi|^2 = 0$, on the boundary. Written in terms of the field h, the two boundary conditions, $|\phi| = 1$ and $\partial_n|\phi|^2 = 0$, would then read, $h = 0$ and $\partial_n h = 0$. Now, away from the zeros of the Higgs field and also very close to the boundary, (2.9) reduces approximately to a Laplace equation. Thus, close to the boundary we are left with a Laplace equation with both Dirichlet and Neumann boundary conditions. This is too strong a constraint for any interesting solution to exist.

Choosing Dirichlet conditions retains the energy quantization property, but the total magnetic flux is no longer quantized. To see this, we rewrite the energy integral (2.6) for Bogomol'nyi vortices as

$$E = \tfrac{1}{2}\int_M [\pm \mathrm{i}\{\partial_2(\bar{\phi}D_1\phi) - \partial_1(\bar{\phi}D_2\phi)\} \pm F_{12}]. \qquad (2.11)$$

In the sector with vortex number N, one can take the phase of the Higgs field on the boundary to be $\mathrm{e}^{\mathrm{i}N\theta}$, $0 \leqslant \theta < 2\pi$. A straightforward computation gives

$$E = \pi N. \qquad (2.12)$$

On the other hand, since $D_t\phi \neq 0$ on the boundary, the tangential component of the gauge potential is not directly related to the winding number. Thus, the total magnetic flux, which is the integral of the tangential component of the gauge potential on the boundary, is no longer quantized.

Before concluding this section, let us briefly point out that the non-vanishing of $D_n\phi$ on the boundary, which is a consequence of Dirichlet boundary conditions, is not uncommon in superconductivity. In the phenomenological Ginzburg–Landau theory for superconductivity, which is the non-relativistic version of the Abelian Higgs model, one usually requires the vanishing of the normal component of the supercurrent on the boundary. The current in the Ginzburg–Landau theory is given by

$$J_i = \frac{\mathrm{i}}{2}(\bar{\phi}D_i\phi - \phi\overline{D_i\phi}), \qquad i = 1, 2. \qquad (2.13)$$

To have $J_n = 0$ on the boundary it is enough to set $D_n\phi = a\phi$, for some real a, on the boundary. Such a choice is usually considered for the type II superconductors and metal/alloy junctions [5].

3. The existence of solutions

The question of existence of Bogomol'nyi vortices has been dealt with in various ways. Taubes [14, 9] used functional analysis in order to prove their existence on $\mathbb{R}^2$ whereas others [4, 8] used differential geometric arguments to prove their existence on a compact manifold without a boundary. Our case, topologically a disk, is closely related to Taubes' work. Taubes' arguments can be applied to our case with only minor changes. Moreover, since our domain is compact, Taubes' arguments simplify greatly. In what follows we will just point out the changes necessary to make Taubes' arguments work in our case. For details the reader is referred to [9].

In this section unless otherwise stated we will take M to be a flat disk of circular boundary of radius R. We can set $\Omega = 1$ in all of the formulae of the previous section.

For a given N zeros of the Higgs field on $\mathbb{R}^2$, Taubes basically proved the existence and uniqueness of the solutions to the equation (2.9) on $\mathbb{R}^2$. We will do the same for M. Let us take the N zeros of the Higgs field to be at $\boldsymbol{x}_i$, $i = 1, \ldots, N$. The first task is to eliminate the singularities on the right-hand side of (2.9). To do this, write h as

$$h = v + u_0 \tag{3.1}$$

such that

$$\triangle u_0 = 4\pi \sum_{i=1}^{N} \delta^2(\boldsymbol{x} - \boldsymbol{x}_i). \tag{3.2}$$

We take the boundary condition for u_0 to be $u_0 = 0$ on the boundary. Since $h = 0$ on the boundary, this means that $v = 0$ on the boundary as well. Such a u_0 is given by the following Green function of the disk

$$u_0(\boldsymbol{x}, \boldsymbol{x}_i) = \sum_{i=1}^{N} \log\left[\frac{R^2|\boldsymbol{x} - \boldsymbol{x}_i|^2}{(R^2 - \boldsymbol{x}^2)(R^2 - \boldsymbol{x}_i^2) + R^2|\boldsymbol{x} - \boldsymbol{x}_i|^2}\right]. \tag{3.3}$$

Note that our choice of u_0 is different from Taubes'. In the notation of Taubes this choice of u_0 makes '$g_0 = 0$'. In terms of v, (2.9) becomes

$$\triangle v = \mathrm{e}^{v+u_0} - 1. \tag{3.4}$$

Thus, in order to deal with the existence problem, one needs to prove the existence of solutions to (3.4). Here, v lives in the space $C_0^\infty(M)$, the space of smooth functions with support on the interior of M.

The strategy to show the existence and uniqueness of solutions to (3.4) is first to define a suitable convex functional and then to prove the existence of a unique minimum of the functional. Such a convex functional is

$$a(v) = \int_M [\tfrac{1}{2}|\nabla v|^2 - v + \mathrm{e}^{u_0}(\mathrm{e}^v - 1)]\,\mathrm{d}^2\boldsymbol{x} \tag{3.5}$$

which is defined on $C_0^\infty(M)$. Again note that this functional is different from Taubes'. The variation of the functional gives (3.4). The domain of the functional $a(v)$ can be extended from $C_0^\infty(M)$ to the Sobolev space H_1, which is the completion of $C_0^\infty(M)$. Proving this assertion, along the line of Taubes', for the flat disk M is rather easy; since due to the compactness of the disk one only needs to worry about the logarithmic singularities of u_0 which are locally square-integrable. For the functional $a(v)$, whose domain is now H_1, one can show the existence of the functional derivatives. Next, following Taubes, it can be shown that (3.5) satisfies similar coercive estimates as in the $\mathbb{R}^2$-case. In order to prove this we may use lemma 3.8 in [9]. There, in the notation of Taubes, one should take '$b = \frac{1}{2}$'. This can be seen by putting $g_0 = 0$ and $\mathrm{e}^{u_0} \leqslant 1$ in the latter part of the proof of that lemma. Using all these facts, the existence of a unique minimum of the functional $a(v)$ is guaranteed (propositions VI.7.7, 7.8 in [9]). Thus, given the positions of N zeros of the Higgs field, there exists a unique solution to the Bogomol'nyi equations for vortices on a flat disk with a circular boundary. We conjecture that in the same way, which could be technically more difficult, one should be able prove the existence of solutions for vortices on a region which is topologically a disk.

The positions of the zeros of the Higgs field parametrize the moduli space. In the given sector of vortex number N the dimension of the moduli space is $2N$. We would like to point out that one might try to compute the dimension of the moduli space using Atiyah–Singer index theory. We will relegate this computation to the appendix. However, using the index theory we obtain a different count for the dimension of the moduli space. This is because, in index theory computation one needs to use global boundary conditions which are not compatible with our choice of Dirichlet boundary conditions.

We should note that for vortices on M, which is topologically a disk, there is one important difference from vortices on a compact region without a boundary, namely that there is no Bradlow's bound [4] as far as the existence of the solutions is concerned. On a compact region without boundary, integrating the Bogomol'nyi equation (2.8) and noting that $|\phi|^2$ is positive, one obtains an upper bound to the first Chern number. This bound is $N \leqslant A/4\pi$, where A is the area of the compact region. Thus, for a given vortex number N, there is a minimal choice for the area of the compact region for the vortex solutions to exist. On the other hand, for Bogomol'nyi vortices on M, there is no such bound. This is because the corresponding Bradlow's bound is expressed in terms of the first Chern number, which is usually a fraction, and is no longer the vortex number. Our numerical calculations in the last section confirm this fact.

4. The moduli space near the boundary

4.1. Disk with a smooth boundary

In this section unless otherwise stated we will take M to be a two-dimensional region which is topolgically a disk and we will use complex coordinates. As mentioned in the introduction, according to Manton [10], the kinetic energy given by (2.3) determines a natural metric on the moduli space. Samols [12] obtained an expression for this metric using data around the zeros of the Higgs field. As his analysis is local, the same formula of the metric can be adapted for the disk M. The only thing to worry about is a contribution to the metric coming from the boundary. However, Dirichlet conditions, $h = 0$ on the boundary, can be used to show that such a contribution is zero. Let $z_i = x_{1i} + \mathrm{i}x_{2i}$, $(i = 1, \ldots, N)$ denote the zeros of the Higgs field. The boundary contribution to the metric is $-\mathrm{i}\int_{\partial M} \mathrm{d}\bar{z}\, \tilde{\eta}\partial_{\bar{z}}\tilde{\eta}$, where $\tilde{\eta} = \sum_{i=1}^{N} \mathrm{d}z_i\, \frac{\partial h}{\partial z_i}$ [12]. Clearly, this boundary contribution to the metric is zero as $\tilde{\eta}$ is zero on the boundary.

Near a simple zero z_i of the Higgs field, h has the following series expansion

$$h = \log|z - z_i|^2 + a_i + \tfrac{1}{2}\{b_i(z - z_i) + \bar{b_i}(\bar{z} - \bar{z_i})\} \\ + c_i(z - z_i)^2 - \frac{\Omega(z_i)}{4}(z - z_i)(\bar{z} - \bar{z_i}) + \bar{c_i}(\bar{z} - \bar{z_i})^2 + \cdots. \tag{4.1}$$

Here, the coefficient $\Omega(z_i)/4$ of the mixed quadratic term is obtained by substituting the above expression in (2.9). The metric of the moduli space is

$$\mathrm{d}s^2 = \sum_{i,j=1}^{N} \left(\Omega(z_i)\delta_{ij} + 2\frac{\partial \bar{b_j}}{\partial z_i} \right) \mathrm{d}z_i\, \mathrm{d}\bar{z_j}. \tag{4.2}$$

The metric has two contributions. The first is induced from the metric of M on which the vortices are moving and the second is a modification to it which we will call the excess metric. The associated Kähler form is

$$\omega = \frac{\mathrm{i}}{2} \sum_{i,j=1}^{n} \left(\Omega(z_i)\delta_{ij} + 2\frac{\partial \bar{b_j}}{\partial z_i} \right) \mathrm{d}z_i \wedge \mathrm{d}\bar{z_j}. \tag{4.3}$$

Then, the volume of the moduli space is

$$\mathrm{Vol}_N = \frac{1}{N!}\int \omega^N. \tag{4.4}$$

For one vortex the volume (area) of the moduli space is

$$\mathrm{Vol}_1 = \frac{\mathrm{i}}{2}\int_M \Omega \mathrm{d}z \wedge \mathrm{d}\bar{z} + \mathrm{i}\int_{\partial M} \bar{b}_1 \,\mathrm{d}\bar{z} \tag{4.5}$$

where the first integral is the area of M and the second integral is a modification to it which we will call the excess volume. It is to be noted that for a single vortex on $\mathbb{R}^2$ the excess volume is zero, since $b_1 = 0$ due to the translational symmetry. In future, for a single vortex we will denote the excess metric and the excess volume of the moduli space by $\mathrm{d}s_1^2$ and V_1 respectively. We will also denote b_1 by b. In the rest of this section we will be concerned with finding b near the boundary. This can be used to obtain the metric of the moduli space near the boundary and the excess volume of the moduli space. Our primary concern will be a one-vortex case, since knowing that will be the first step to obtaining the metric for the N vortex moduli space which is topologically the symmetrized Nth power of the one-vortex moduli space.

We have already mentioned that near the boundary, due to Dirichlet boundary conditions, it is possible to approximate the nonlinear equation (2.9) by a Laplace equation with a source. Heuristically, this can be seen as follows. There are two distance scales involved in the problem. One is the size of the disk and the other is the distance of the vortex from the origin. Near the boundary it may appear that one can forget about the exponential term in (2.9), but the nonlinear nature of the equation may not permit this. The effect of nonlinearity can be unravelled by noting that (2.9) can be solved by an iteration of the following form [1]

$$h = h_0 + h_n \tag{4.6}$$

where

$$\triangle h_0 = 4\pi\delta^2(z - z_1) \tag{4.7}$$

with $h_0 = 0$ on the boundary and

$$\triangle h_n = \Omega(\mathrm{e}^{h_{n-1}} - 1) \tag{4.8}$$

with $h_n = 0$ on the boundary. It is amusing to note the absence of Ω in (4.7). h_0 is the Green function of the disk M. In terms of h_0, the solution of h_n is

$$h_n(z, z_1) = \frac{1}{4\pi}\int_M \Omega(z')h_0(z, z')(\mathrm{e}^{h_{n-1}(z_1,z')} - 1)\,\mathrm{d}z'\,\mathrm{d}\bar{z}'. \tag{4.9}$$

The iterative solution h_n is smooth and moreover, it can be shown that it converges to a unique limit [1]. Now, if z is near the boundary then irrespective of the position of the source $|h_n(z, z_1)| \sim \mathrm{O}(\epsilon')$. Here ϵ', a small positive number, is the distance between z and the nearest point on the boundary. On the other hand, assuming that the source is not near the boundary, one can estimate using the Green function of M that $h_0 \sim h_n$ near the boundary. In order to have $h_0 \sim \mathrm{O}(1)$ near the boundary, so that the nonlinear correction h_n can be neglected, we require $|z - z_1| \sim \epsilon'$. Thus, to use a Laplace equation with a source as an approximation to the nonlinear equation (2.9), the source as well as the point we are looking at should lie quite close to the boundary. However, this is quite well suited to our purpose, since in order to evaluate the volume of the moduli space we need to know b while keeping the vortex always a small fixed distance away from the boundary.

We will now solve (4.7). The vortex is at a distance ϵ, say, away from the boundary and we want to solve a Laplace equation with a source with the correct boundary value, for length up to $\mathrm{O}(\epsilon)$ along the boundary (i.e. the part of the boundary nearest to the vortex). In such a case, the method of images can be applied. For generality, we consider the boundary of M to be given by a closed curve. We will first linearly approximate the boundary by its

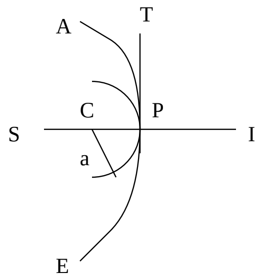

Figure 1. Geometry of the boundary at an arbitrary point.

tangent and next we will quadratically approximate it by taking curvature into consideration. Let APE be a section of the boundary and let the position of the source, S, be at $\xi = \xi_1 + i\xi_2$, and SP be the normal to the boundary with P on the boundary being at $\xi_P = \xi_{1P} + i\xi_{2P}$ (figure 1). The image, I, is at $\xi_I = \xi_{1I} + i\xi_{2I}$. PI is normal to the boundary at P, and SP = PI, in the linear approximation.

Then, h_0 is sum of the potentials due to the source, S, and the image, I,

$$h_0 = \log|z-\xi|^2 - \log|z-\xi_I|^2. \tag{4.10}$$

From this, we find that

$$b = (\xi_P - \xi)^{-1}. \tag{4.11}$$

Let us consider the vortex restricted to being more than distance ϵ away from the boundary. The leading contribution to the volume (area) of the moduli space comes from the excess volume and the next subleading contribution (up to an O(ϵ) ambiguity) is given by the actual area of the disk. The excess volume of the moduli space is

$$V_1 = \frac{\mathcal{S}}{\epsilon} \tag{4.12}$$

where $\mathcal{S}$ is the perimeter of the boundary and $\epsilon = |\xi - \xi_P|$ = SP. The excess metric of the moduli space near the boundary is

$$ds_1^2 = \frac{2}{(\bar{\xi}_P - \bar{\xi})^2}\frac{\partial\bar{\xi}_P}{\partial\xi}[d\xi_1^2 + d\xi_2^2]. \tag{4.13}$$

From (4.12) it can be seen that the excess volume of the moduli space diverges as one approaches the boundary. In fact we will see shortly that near the boundary, apart from a constant factor, the metric is that of a hyperbolic plane. This divergence of the volume is due to Dirichlet boundary conditions. Actually, very close to the boundary the magnitude of the Higgs field has to rise sharply from 0 to 1. This makes the derivative and, hence, the volume divergent.

In terms of the normal to the boundary, $\boldsymbol{N} = \hat{e}_{x_1}N_1 + \hat{e}_{x_2}N_2$, at P, $\bar{b}$ can be written as $\bar{b} = (N_1 + iN_2)/\epsilon$. This can be used to obtain

$$\frac{\partial\bar{b}}{\partial\xi} = \left|\nabla\cdot\left(\frac{\boldsymbol{N}}{2\epsilon}\right)\right| + i\left|\nabla\times\left(\frac{\boldsymbol{N}}{2\epsilon}\right)\right|. \tag{4.14}$$

Now, let us go to a coordinate system such that its origin coincides with the position of the vortex. Then, $N_2 = 0$ and $N_1 = -1$ which means that $\xi_1 = \xi_2 = \xi_{2P} = 0$ and $\xi_{1P} = -\epsilon$.

Equation (4.14) gives $\frac{\partial \bar{b}}{\partial \xi} = 1/2\epsilon^2$. Using (4.13), we obtain the following excess metric near the boundary

$$\mathrm{d}s_1^2 = \frac{1}{\epsilon^2}(\mathrm{d}\epsilon^2 + \mathrm{d}\xi_2^2). \tag{4.15}$$

Clearly, this is the metric of a hyperbolic plane.

For a flat disk with a circular boundary of radius R, one can exactly solve a Laplace equation with a source. This can be compared with the above results obtained by the linear approximation where the boundary is taken to be of arbitrary shape. First, one finds that $\xi_{\mathrm{P}} = R\xi/|\xi|$. Then, in the linear approximation, using (4.5), the excess volume is

$$V_1 = (2\pi R)/\epsilon, \tag{4.16}$$

and from (4.13), the excess metric near the boundary is

$$\mathrm{d}s_1^2 = ((R - r))^{-2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2). \tag{4.17}$$

In the coordinate system used in the previous paragraph one obtains

$$\mathrm{d}s_1^2 = \frac{1}{\epsilon^2}(\mathrm{d}\epsilon^2 + \mathrm{d}\xi_2^2). \tag{4.18}$$

On the other hand, using the exact Green function for a flat disk, (3.3) with $N = 1$, we find that

$$b = 2\bar{\xi}/(R^2 - |\xi|^2). \tag{4.19}$$

Thus, when the vortex is near the boundary, one obtains the excess volume

$$V_1 = (2\pi R)/\epsilon \tag{4.20}$$

which is divergent. Note that this is the same as (4.16). The excess metric is

$$\mathrm{d}s_1^2 = 4r^2(R^2 - r^2)^{-2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) \tag{4.21}$$

which reduces to (4.17) when $r \sim R$.

Now, we will take into account the curvature of the boundary. Here, we take the source, S, to be at ξ with SP $= \epsilon$. Let a_{P} and c_{P} be, respectively, the radius and the centre of curvature, C, of the boundary at P (figure 1). Then, the solution of (4.7), such that it satisfies the boundary condition for a boundary length $\mathrm{O}(\epsilon)$, is

$$h_0 = \log\left[\frac{a_{\mathrm{P}}^2|z - \xi|^2}{(|z - c_{\mathrm{P}}|^2 - a_{\mathrm{P}}^2)(|\xi - c_{\mathrm{P}}|^2 - a_{\mathrm{P}}^2) + a_{\mathrm{P}}^2|z - \xi|^2}\right]. \tag{4.22}$$

This gives

$$b = \frac{2(\bar{\xi} - \bar{c_{\mathrm{P}}})}{a_{\mathrm{P}}^2 - |\xi - c_{\mathrm{P}}|^2}. \tag{4.23}$$

If, as before $|\xi_{\mathrm{P}} - \xi| = \epsilon$, i.e. the vortex is quite close to the boundary, we obtain

$$b = \frac{2(\bar{\xi} - \bar{c_{\mathrm{P}}})}{\epsilon(2|\xi - c_{\mathrm{P}}| + \epsilon)}. \tag{4.24}$$

After a lengthy calculation one can show that the excess volume is

$$V_1 = \frac{\mathcal{S}}{\epsilon} - 3\pi \tag{4.25}$$

where $\mathcal{S}$ is the perimeter of the boundary. The volume still diverges. At $\mathrm{O}(1/\epsilon)$, both the linear, (4.12), and the quadratic approximation, (4.25), give the same result for the excess volume. Going to a special coordinate system with $c_{\mathrm{P}} = 0$, at $\mathrm{O}(1/\epsilon)$, b of (4.23) and (4.11) agree. Thus, the hyperbolic nature of the metric emerges here also. This means that the divergence of the volume of the moduli space and the appearance of the hyperbolic metric near the boundary is a generic feature for a single vortex moving on a two-dimensional region which topologically looks like a disk. Further, these properties are independent of the shape of the boundary.

Let us briefly comment on the the constant term in the expression (4.25). Such a constant term cannot be obtained by adding the nonlinear correction term h_n to h_0. Its appearance initially seems to have the following inconsistency. As we shrink the disk size by letting $a_{\mathrm{P}} \sim \epsilon$ and $\epsilon \to 0$ so that the vortex cannot move, one obtains $V_1 = -\pi$. This contradicts the expectation that when $a_{\mathrm{P}} = 0$, the excess volume (also, the total volume) is zero. However, in this case (4.25) is no longer valid. This is because to obtain (4.25) we used (4.24) which is obtained by assuming that both a_{P} and $|\xi| \gg \epsilon$. In the case when the disk radius goes to zero one should consider the exact expression (4.23) for b. Hence, taking the limit when both $a_{\mathrm{P}} \to 0$ and $|\xi| \to 0$, one finds that $b = 0$. Thus, the excess volume is zero as expected.

4.2. Region with corners

We will briefly discuss the case when the vortex moves on a rectangle or region with corners, since the numerical study of the next section will involve vortices on a square. The linear approximation works here as well, except at the corners.

Let us consider a vortex on a flat rectangle. We will see that at $O(1/\epsilon)$ the excess volume of the moduli space still agrees with the previous results. For a rectangle of width a and length l (figure 2), a single source inside the rectangle produces an infinite lattice of images. It can be shown that the solution of (4.7) is

$$h_0 = \frac{1}{2} \log \left| \frac{\mathcal{P}(z-\bar{\xi})\mathcal{P}(z+\bar{\xi})}{\mathcal{P}(z-\xi)\mathcal{P}(z+\xi)} \right|^2 \tag{4.26}$$

where $\mathcal{P}(\ )$ denotes Weierstrass' $\mathcal{P}$-function with the complex period $2a + \mathrm{i}2l$. It is clear from (4.26) that h_0 vanishes on the edges of the rectangle. The excess volume is

$$V_1 = (2a + 2l - 8\epsilon)/\epsilon = \mathcal{S}/\epsilon \tag{4.27}$$

as expected.

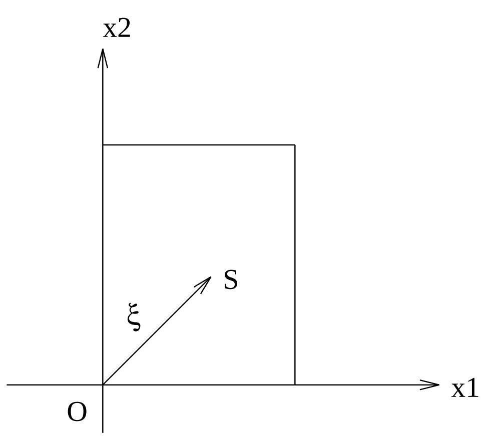

Figure 2. One vortex inside a rectangle.

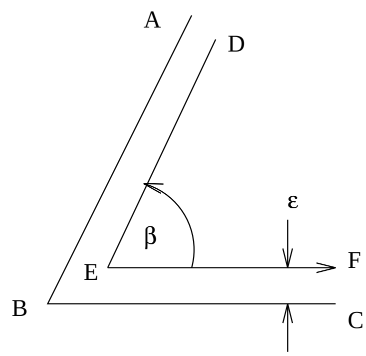

Figure 3. Geometry around a corner.

Now, we will work out the changes in the formulae for the excess volume and the excess metric for a vortex moving around a corner of arbitrary opening angle. These will be used for comparison with the numerical result of the next section. Let us first find the change in the excess volume. Suppose that the corner, ABC, with opening angle β, is situated at the origin and one of its edges lies on the x_1-axis (figure 3). The source, S, is at ξ. Then

$$h_0 = \log \left| \frac{z^{\pi/\beta} - \xi^{\pi/\beta}}{\bar{z}^{\pi/\beta} - \xi^{\pi/\beta}} \right|^2 \tag{4.28}$$

which gives

$$b = (\pi/\beta - 1)/\xi - \frac{2\pi}{\beta} \frac{\xi^{\pi/\beta - 1}}{\xi^{\pi/\beta} - \bar{\xi}^{\pi/\beta}}. \tag{4.29}$$

Let us consider that, around the corner the vortex traverses a symmetrically located path of length 2ϵ (DEF in figure 3). Then, the contribution from the corner to the excess volume is

$$\delta V_1 = \frac{(2\epsilon)}{\epsilon} - 2(1 - \pi/\beta)(\alpha_1 - \alpha_2) \tag{4.30}$$

where α_1, α_2 are, respectively, the phases of the initial and the final position of the vortex. These are, $\alpha_1 = \tan^{-1}(1 + \cot\beta/2)$ and $\alpha_2 = \beta - \alpha_1$. The first term in (4.30) is from a corresponding region with the corner being rounded off and the second term is solely a corner effect. Once again the excess volume is divergent. Similarly, the excess metric around the corner is

$$\mathrm{d}s_1^2 = 4(\pi/\beta)^2 \frac{(\xi\bar{\xi})^{\pi/\beta - 1}}{(\bar{\xi}^{\pi/\beta} - \xi^{\pi/\beta})^2} (\mathrm{d}\xi_1^2 + \mathrm{d}\xi_2^2). \tag{4.31}$$

4.3. An example of a closed and circular geodesic

The hyperbolic nature of the metric near the boundary of the one-vortex moduli space has motivated us to look into this. Let us consider a vortex on a flat disk M' with an outer circular boundary of radius R. Further, we take M' to contain a hole of radius R'. The one-vortex moduli space is topologically M'. The circularly symmetric nature of the problem suggests that the most general form of the metric of one-vortex moduli space is

$$\mathrm{d}s^2 = \rho(r)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2). \tag{4.32}$$

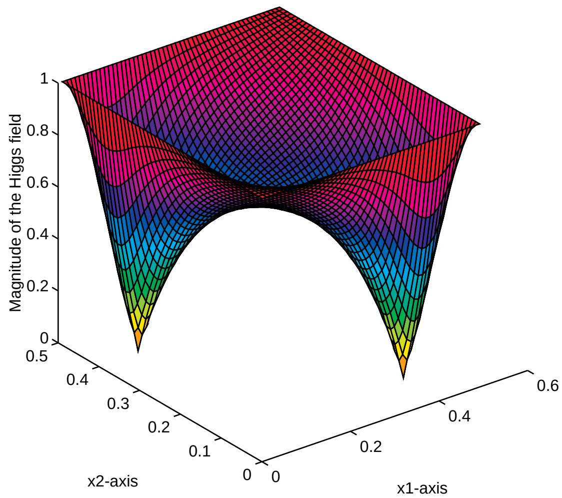

Figure 4. Two-vortex solution on a square of size 0.5.

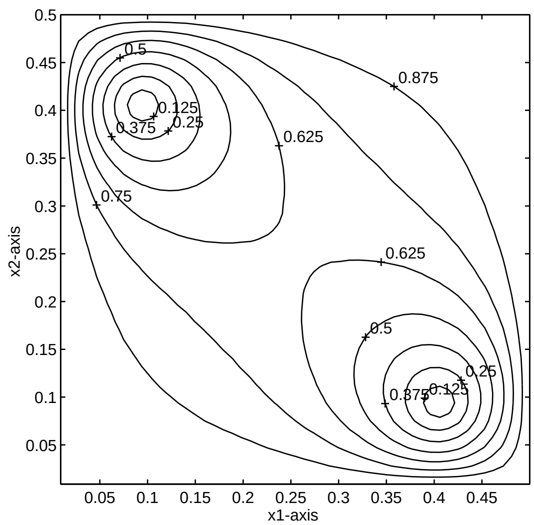

Figure 5. Contour plot of the Higgs field for two vortices on a square of size 0.5.

From our previous analysis we know that the vortex moduli space looks like a hyperbolic plane near the outer boundary. The same result also holds true near the inner boundary. Thus, near the two boundaries, respectively, at $r = R$ and $r = R'$, we obtain, using (4.21)

$$\rho \sim \frac{4R^2}{(R^2 - r^2)^2} \qquad \text{as } r \to R \tag{4.33}$$

and

$$\rho \sim \frac{4R'^2}{(r^2 - R'^2)^2} \qquad \text{as } r \to R'. \tag{4.34}$$

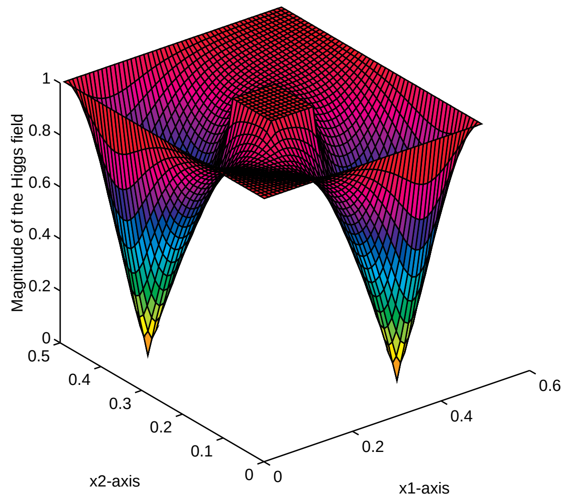

Figure 6. Two-vortex solution on a square of size 0.5 with the hole size 0.1.

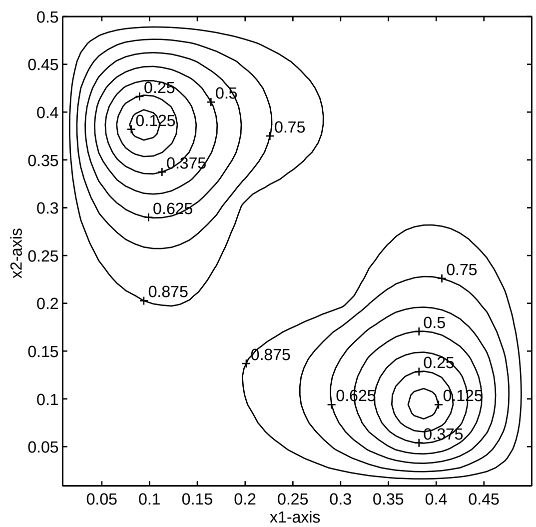

Figure 7. Contour plot of the Higgs field for two vortices on a square of size 0.5 with the hole size 0.1.

Now, if $r = r(\theta)$ denotes a geodesic in the moduli space, the equation of the geodesic is

$$r'' - \left(\frac{1}{r} + \frac{\psi(r)}{2}\right) r'^2 - \frac{r^2}{2}\psi(r) = 0 \tag{4.35}$$

where $r' = \mathrm{d}r/\mathrm{d}\theta$, $r'' = \mathrm{d}^2r/\mathrm{d}\theta^2$ and $\psi = \mathrm{d}(\ln(\rho r^2))/\mathrm{d}r$. From (4.35) one sees that at each zero of $\psi(r)$ there exists a closed circular geodesic. Equations (4.33) and (4.34) imply that ψ is negative near $r \sim R'$ and ψ is positive near $r \sim R$. Hence, there must be at least one zero of ψ at $r = R_0$, say. This proves the existence of a closed, circular geodesic at $r = R_0$ in the moduli space of one vortex on M'. This closed, circular geodesic will be unique if ψ has a unique zero. If there is such a unique geodesic it may be plausible that

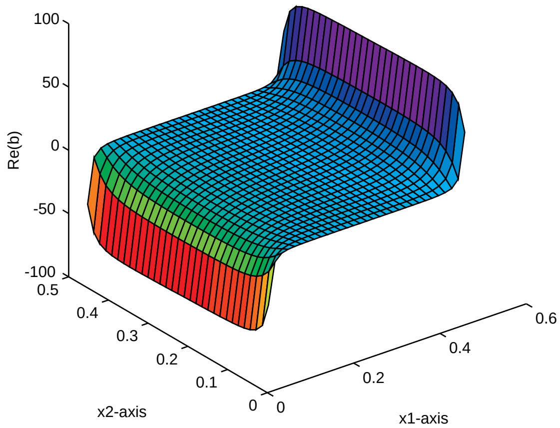

Figure 8. Real part of b as a function of vortex position, for one vortex on a square of size 0.5.

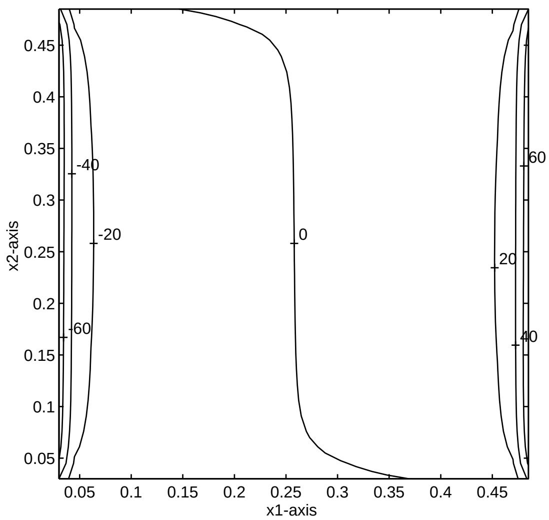

Figure 9. Contour plot of $\mathrm{Re}\,(b)$ for one vortex on a square of size 0.5.

this will be unstable. By this we mean that any nearby geodesic will always hit either one of the two boundaries.

5. Numerical results

In this section we will verify numerically some of the analytical results obtained in the previous sections. We will set $\Omega = 1$ throughout this section. To check if there are solutions to the Bogomol'nyi equations with Dirichlet boundary conditions, we consider (2.9) to be solved by numerical methods. For computational convenience, we first rewrite h in a slightly different way, as $h = f + u_1$, where

$$u_1 = \sum_{i=1}^{N} \log |\boldsymbol{x} - \boldsymbol{x}_i|^2. \tag{5.1}$$

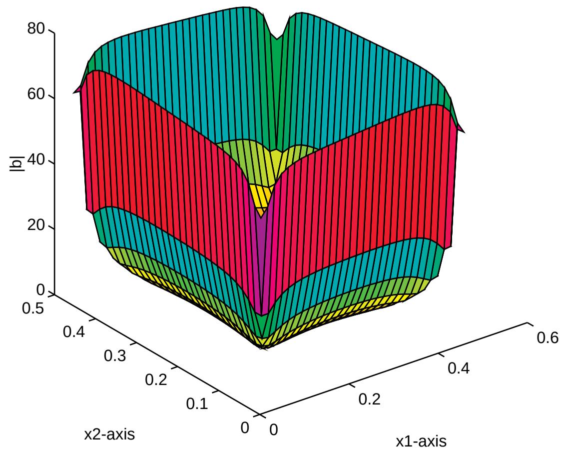

Figure 10. Magnitude of b for one vortex on a square of size 0.5.

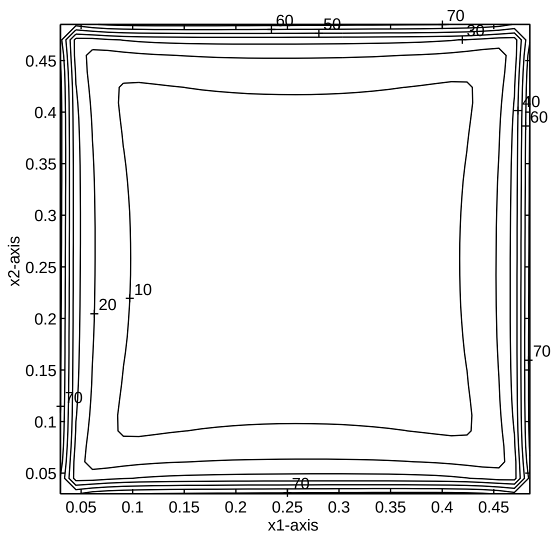

Figure 11. Contour plot of $|b|$ for one vortex on a square of size 0.5.

Then, the equation for f is

$$\triangle f = \mathrm{e}^{f+u_1} - 1 \tag{5.2}$$

with the boundary condition $f|_{\partial M} = -u_1|_{\partial M}$. In terms of f, the modulus of the Higgs field and the coefficients b_i's in the expansion of h are, respectively, $|\phi| = \mathrm{e}^{f+u_1}$ and $b_i = \partial f/\partial z|_{z=z_i}$, $i = 1, \ldots, N$. To compute the energy of the vortex configuration we rewrite (2.6) using the Bogomol'nyi equations, in the following way

$$E = \int_M \mathrm{d}^2 x \, [(\partial_1 |\phi|)^2 + (\partial_2 |\phi|)^2 + \tfrac{1}{4}(1 - |\phi|^2)^2] \tag{5.3}$$

which is more amenable to numerical computation. The elliptic equation (5.2) can be solved numerically using the successive over relaxation (SOR) method [7].

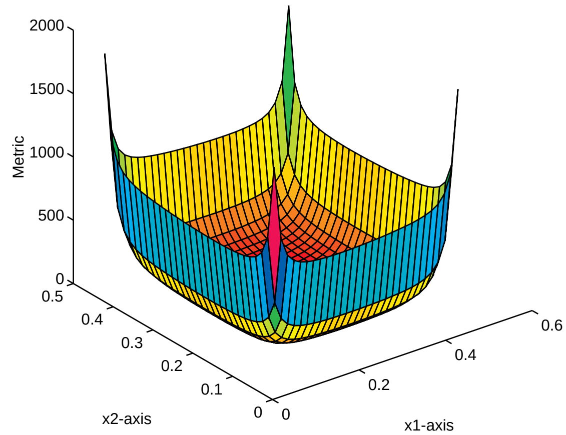

Figure 12. Moduli space metric (i.e. the coefficient of $dz\,d\bar{z}$) for one vortex on a square of size 0.5.

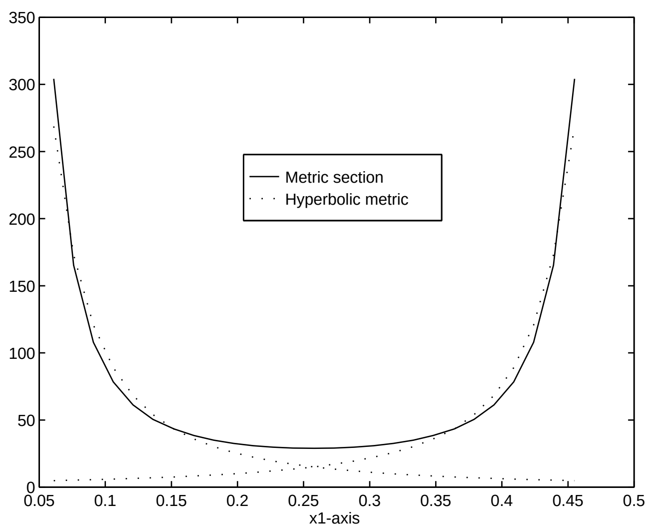

Figure 13. A section of the metric across the middle of the moduli space and sections of the hyperbolic metric.

For computational ease, instead of taking a disk we considered vortices on a square domain. The square domains we took were of various sizes, namely 0.5, 1, 1.5 units which were discretized into meshes of sizes 33×33, 43×43, and 53×53, respectively. We found that (5.2) relaxes to give vortex solutions by taking $f = 0$ inside the square and the required boundary value on the edges (figures 4 and 5). The tolerance for the residual norm was taken to be 10^{-4}. It was found that for the square of size 0.5, about 100 sweeps were required for the relaxation to reach below the tolerance level and the number of sweeps varied linearly with the size of the square. Concerning whether these solutions really describe Bogomol'nyi vortices, we also computed the energy and the total magnetic flux. It was found that the energy varied linearly with the vortex number. Also, the energy was independent of the size of the square and of the positions of the vortices. As an example, for the square of size

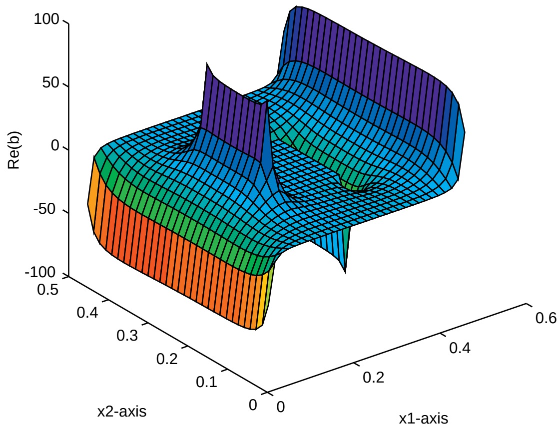

Figure 14. Real part of b as a function of vortex position, for one vortex on a square of size 0.5 with the hole size 0.1.

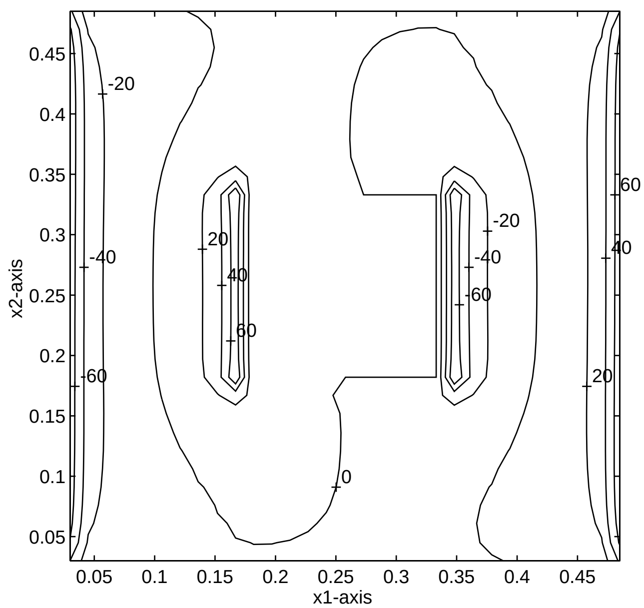

Figure 15. Contour plot of Re$\,(b)$ for one vortex on a square with the hole size 0.1.

0.5, and $N = 1$ the energy was found to be 3.144. This is pretty close to the theoretical value, $E = \pi$. On the other hand, the total magnetic flux varied with the size of the square, and also, with the position of the vortices. However, for a given vortex number, the total magnetic flux did not vary much when the size of the square was sufficiently large. This is consistent with the fact that in the limit when the disk size is infinite the total magnetic flux is constant. To check the dimension of the moduli space we considered all of the squares and put two vortices in various positions. In one case they were only one grid point away from the boundary. Vortex solutions were found in all cases. Moreover, the existence of solutions in these cases means that there is no Bradlow's bound as such.

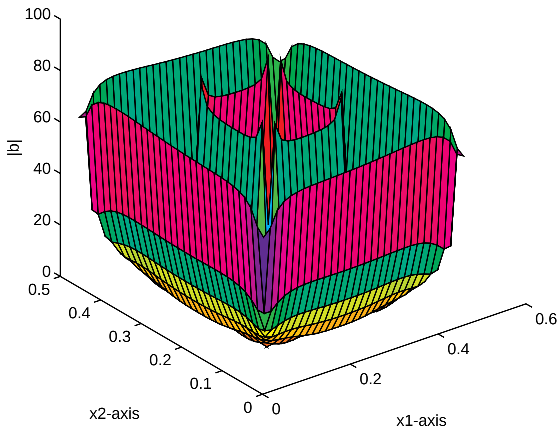

Figure 16. Magnitude of b for one vortex on a square of size 0.5 with the hole size 0.1.

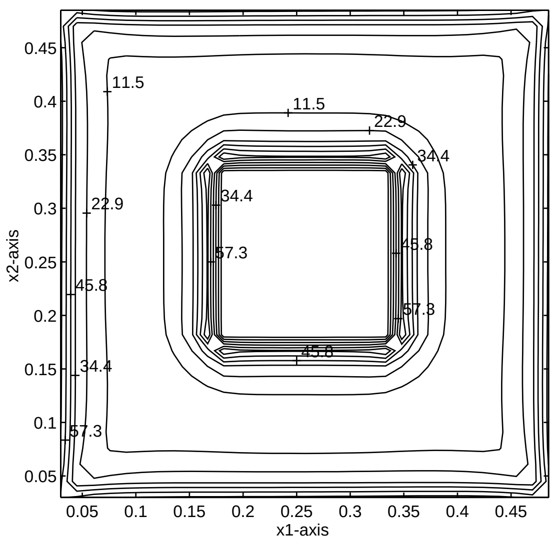

Figure 17. Contour plot of $|b|$ for one vortex on a square of size 0.5 with the hole size 0.1.

Next, we carried over the whole procedure onto squares with square holes centred at the origin. For the inner boundary we took the same boundary condition as the outside one. Solutions were found in these cases as well, with energy varying with the vortex number only and, the total magnetic flux likewise not a constant. One interesting feature in this case is that vortices are more localized—the Higgs field taking the vacuum expectation value almost everywhere (see e.g. figures 6 and 7)—but, as we decreased the size of the hole, the Higgs field was more spread out in space with the field configuration approaching the no-hole case.

The excess volume of the moduli space was computed in the one-vortex case. We kept the vortex one grid point away from the boundary. This computation has some interesting features. First, there is nice symmetry in the plot of the real and imaginary parts of b. The plot and the contour plot of the real part of b are shown in figures 8 and 9, respectively. The corresponding plots of the imaginary part of b can be obtained by a 90° rotation of figures 8 and 9. Second, the imaginary part in the formula (4.5) for the excess volume was found to be zero as it should be since the volume is a real number. We found quite good agreement between the numerical and the analytical result of the excess volume. As an example, for the square of size 0.5, divided into 33×33 grid points, the $O(1/\epsilon)$ contribution to the excess volume is 129.22. On the other hand the corresponding result found numerically was 129.62. To get some feeling about the size of the volume, we also computed the magnitude of b (figures 10 and 11). Close to the boundary, b takes very large values. From the contour plot of b, figure 11, one can see that the magnitude of b is fairly constant around the edges except at the corners. A close inspection of these figures reveal that there are glitches around the corners. There the drop in the value of b is about 30%. It is in close agreement with the analytical result (4.29) (opening angle $\beta = \pi/2$). Moreover, such corner effects are concentrated in regions $O(\epsilon)$. This validates the assumption we made in the corner analysis of the previous section. We also computed the metric of the moduli space (figure 12). To explore the hyperbolic nature of the metric, we took a section of the metric across the middle of the square, and superimposed on it the corresponding sections of the hyperbolic metric (figure 13).

Finally, the excess volume and the various components of b for the squares with holes were computed (figures 14–17). There also, we found quite good agreement between the numerical and the analytical result as long as the hole size is relatively small compared with the square size. In one case with the square (size = 0.5, hole size = 0.1) divided into 33×33 grid points, the excess volume was found to be equal to $(128.3 + 51.7)$. The first and the second terms in the bracket are, respectively, due to the outer and the inner boundaries. This is in good agreement with our analytical result. It appears that the two boundaries do not interfere with each other. Actually this is so, as long as the hole size is small relative to the size of the square. When the hole was taken to be comparable to the square itself, the numerical value of the excess volume ceased to agree with the analytical result. In these cases the boundaries interfere in a complicated way; these effects were not taken into consideration in our simple approximation.

Acknowledgments

I am very much indebted to my supervisor Dr N S Manton for suggesting this problem to me and for his continuous guidance, and enlightening discussions that I have had with him during the course of this work. I am also grateful to Dr H Merabet for his generous help in computing. Many thanks to Mr C Houghton and Mr P Irwin for their critical comments on this manuscript. This work was supported by the Overseas Research Scheme, the Cambridge Commonwealth Trust and Wolfson college.

Appendix. Index theory and the dimension of the moduli space

A.1. The dimension of the moduli space

The computation of the dimension of the moduli space using index theory [15] is quicker than other means. In this appendix we first use Atiyah–Singer index theory [2] to compute the dimension in the case of a compact manifold X, without a boundary. Then, we do the same for a compact manifold with a boundary. We shall see how the global boundary conditions [2], which are not compatible with the Dirichlet boundary conditions we are dealing with, changes the result. Finally, in order to appreciate the role of the global boundary conditions we compute the contribution to the index of the dominant elliptic operator by counting the number of the zero modes.

Perturbing the Bogomol'nyi equations (2.7), (2.8) around a solution (A_i, ϕ) to $(A_i + \delta A_i, \phi + \delta\phi)$ we obtain the following linearized equations,

$$\partial_1 \delta A_2 - \partial_2 \delta A_1 + \frac{\Omega}{2}(\phi\delta\bar{\phi} + \bar{\phi}\delta\phi) = 0 \tag{A.1}$$

$$\bar{\partial}_A \delta\phi - \mathrm{i}(\delta A_1 + \mathrm{i}\delta A_2)\phi = 0 \tag{A.2}$$

where $\bar{\partial}_A = D_1 + \mathrm{i}D_2$. The gauge transformation $(A_i, \phi) \to (A_i - \mathrm{i}\partial_i\lambda, \phi + \mathrm{i}\lambda\phi)$ satisfies the above equations. Now, the following elliptic complex can be defined

$$0 \to \Lambda^0(X) \xrightarrow{d} \Lambda^1(X) \oplus \Gamma(X, L_c) \xrightarrow{\tau} \Lambda^2(X) \oplus \Gamma(X, L_c) \to 0 \tag{A.3}$$

where $\tau = (d + \bar{\partial}_A)$, Λ^i is the space of the i-forms on X and $\Gamma(X, L_c)$ is the space of the sections, ϕ, of the line bundle L_c. In terms of the elliptic operator $D = \tau + d^\star$, the above complex can be written as

$$D : \Lambda^1(X) \oplus \Gamma(X, L_c) \to \Lambda^0(X) \oplus \Lambda^2(X) \oplus \Gamma(X, L_c). \tag{A.4}$$

The index of D is then $\mathrm{Ind}\, D = \mathrm{Ind}(d + d^\star + \bar{\partial}_A)$. Now, $\mathrm{Ind}\,\bar{\partial}_A = 2\int_X \mathrm{Ch}(\phi)\mathrm{td}(X)$ [6] where $\mathrm{td}(X)$ is the Todd class of X, $\mathrm{Ch}(L_c)$ is the Chern character of the line bundle L_c, and the factor of 2 is there due to the consideration of the line bundle over the real numbers. In two dimensions, keeping only zero- and two-dimensional cohomology classes in the expansion of the Chern character and the Todd class, we obtain, $\mathrm{Ind}\,\bar{\partial}_A = \chi(X) + 2\int_X c_1(\phi)$. Here, $\chi(X)$ is the Euler characteristic of X [6]. On the other hand, $\mathrm{Ind}(d + d^\star)$ is $-\chi(X)$. Summing up, the index of D, $\dim(\ker D) - \dim(\mathrm{coker}\, D)$, is $2\int_X c_1(\phi)$.

It can be shown that $\dim(\mathrm{coker}\, D)$ is 0 by noting that $DD^\star$ is a positive operator (assuming the first Chern number is positive). Hence, the dimension of the moduli space, $\dim(\ker D)$, is given by $\mathrm{Ind}\, D$. It is just twice the first Chern number. Thus, in the sector with vortex number N, the dimension is $2N$.

A.2. Contribution from the boundary

For the disk M with a boundary the above result receives two more corrections of the following form [6]. We will take M to be a flat disk with a circular boundary of radius R. Let the elliptic operator D be expressed near the boundary as $D = \frac{\partial}{\partial n} + B$, where $\partial/\partial n$ is the normal derivative on the boundary. Then,

$$\mathrm{Index}\, D = V[M] + S[\partial M] + \xi[\partial M]. \tag{A.5}$$

Here, $V[M]$ is a contribution from the bulk. $S[\partial M]$ is a boundary contribution expressed as an integral of a Chern–Simons form. Finally, $\xi[\partial M]$ expresses the spectral asymmetry of the boundary operator B. It is equal to $-(q + \eta(0))/2$ where q is the number of the zero modes of B and $\eta(0)$ is the analytic continuation of $\eta(s)$ to $s = 0$; $\eta(s)$ can be written as, $\eta(s) = \sum_{\lambda_i \neq 0} \mathrm{sign}(\lambda_i)/|\lambda_i|^s$, where λ_i's are the eigenvalues of B.

Let us evaluate the above contributions for a disk. We take the first Chern number c, to lie in the range $t < c \leqslant t + 1$, where t is an integer. By a suitable gauge transformation it is always possible to make $A_\theta = c$ on the boundary. Only the operator $\bar{\partial}_A$ contributes to $V[M]$. We obtain $V[M] = 2c$. $S[\partial M]$ gets contributions from $(d + d^\star)$ only. One can show that $S[\partial M] = \chi(M) = 1$. For a disk of radius R the boundary operator B is $-\frac{\mathrm{e}^{\mathrm{i}\theta}}{R}(\mathrm{i}\partial_\theta + A_\theta)$. Now, the eigenvalues of the boundary operator, B, are simply $\lambda_m = m - c$ with the corresponding eigenfunctions $\mathrm{e}^{\mathrm{i}m\theta}$, $m \in \mathbb{Z}$. After a straightforward computation, using the properties of the Riemann zeta function [16], it can be shown that $\eta(0) = 2(c - t) - 1$. Finally, using (A.5), the index of D is $2(t + 1)$. Here, one should bear in mind a factor of 2 in the computation of $\xi[\partial M]$ which has the same origin as in the computation of the index on the bulk. Thus, the index is determined by the next nearest integer to the first Chern number. Unfortunately, this is not a topologically invariant quantity. This is clearly evident from the fact that as one changes the radius of the disk the tangential component of the gauge potential on the boundary also changes. This changes the index. The reason for getting this topologically non-invariant result is due to the fact that in Atiyah–Singer index theory the boundary operator, B, is assumed to satisfy some kind of global boundary conditions. These global boundary conditions clash with the Dirichlet conditions. For a disk, its role can be appreciated by computing directly the number of the zero modes of the operator $\bar{\partial}_A$ as shown below. Moreover, the computation of the zero modes can be regarded as an independent check to our index calculation.

A.3. Counting zero modes

Here, we will count the number of zero modes of the operator $\bar{\partial}_A$ in a background where the gauge potential is circularly symmetric. We will work in the radial gauge $A_r = 0$. It is possible to choose $A_\theta(0) = 0$. With this choice we are then required to solve the following equation

$$\bar{\partial}_A l = \mathrm{e}^{\mathrm{i}\theta}\left[\partial_r + \frac{1}{r}(\mathrm{i}\partial_\theta + A_\theta(r))\right] l = 0. \tag{A.6}$$

Expanding l in a Fourier series as $l = \sum_n g_n(r)\mathrm{e}^{\mathrm{i}n\theta}$, $n \in \mathbb{Z}$ and substituting into the above equation, we find $g_n = r^n \exp(-\int_0^R \mathrm{d}r\, A_\theta(r)/r)$. On the boundary $l(R) = \sum_n g_n(R)\mathrm{e}^{\mathrm{i}n\theta}$. The global boundary conditions mean in this case that, on the boundary, the projection of $l(R)$ onto the space of the eigenfunctions corresponding to the positive eigenvalues of the boundary operator B is zero. This means that $g_n = 0$ for $n \geqslant t + 1$. Further, demanding that the modes are normalizable, we obtain $g_n \neq 0$ only if $0 \leqslant n \leqslant t$. Thus, $\mathrm{Ker}(\bar{\partial}_A)$, the number of the zero modes (over the real numbers) of the operator $\bar{\partial}_A$, is $2t + 1$. On the other hand, to find the zero modes of the the adjoint operator we need to solve

$$\bar{\partial}_A^\star l' = \mathrm{e}^{-\mathrm{i}\theta}\left[-\partial_r + \frac{1}{r}(\mathrm{i}\partial_\theta + A_\theta(r))\right] l' = 0. \tag{A.7}$$

Expanding l' in a Fourier series as $l' = \sum_n d_n(r)\mathrm{e}^{\mathrm{i}n\theta}$, $n \in \mathbb{Z}$ and substituting into the above equation we obtain $d_n = r^{-n} \exp(\int_0^R \mathrm{d}r\, A_\theta(r)/r)$. The global boundary condition gives $d_n = 0$ for $n \leqslant t$. The condition that these modes are normalizable gives $d_n = 0$ for $n > t + 1$. Hence, $\mathrm{Ker}(\bar{\partial}_A^\star) = 0$. Thus, $\mathrm{Ind}\, \bar{\partial}_A = 2t + 1$.

References

[1] Ames W F 1992 *Numerical Methods for Partial Differential Equation* (Boston, MA: Academic)
[2] Atiyah M F, Patodi V K and Singer I M 1975 *Math. Proc. Camb. Phil. Soc.* **77** 43
[3] Bogomol'nyi E B 1976 *Sov. J. Nucl. Phys.* **24** 449
[4] Bradlow S 1990 *Commun. Math. Phys.* **135** 1
[5] deGennes P G 1966 *Superconductivity of Metals and Alloys* (New York: Addison-Wesley)
[6] Eguchi T, Gilkey P B and Hanson A J 1980 *Phys. Rep.* **66** 213
[7] Flannery B P *et al* 1986 *Numerical Recipes* (Cambridge: Cambridge University Press)
[8] García-Prada O 1994 *Bull. Lond. Math. Soc.* **26** 88
[9] Jaffe A and Taubes C H 1980 *Vortices and Monopoles* (Boston, MA: Birkhäuser)

[10] Manton N S 1982 *Phys. Lett.* **110B** 54
[11] Manton N S 1993 *Nucl. Phys.* B **400** 624
[12] Samols T M 1992 *Commun. Math. Phys.* **145** 149
Samols T M 1990 *PhD Thesis* Cambridge University, unpublished
[13] Shah P A and Manton N S 1994 *J. Math. Phys.* **35** 3
[14] Taubes C H 1980 *Commun. Math. Phys.* **72** 277
[15] Weinberg E J 1979 *Phys. Rev.* D **19** 3008
[16] Whittaker E T and Watson G N 1920 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)