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Volume of Vortex Moduli Spaces

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Abstract: A gas of N Bogomol'nyi vortices in the Abelian Higgs model is studied on a compact Riemann surface of genus g and area A. The volume of the moduli space is computed and found to depend on $N,\ g$ and A, but not on other details of the shape of the surface. The volume is then used to find the thermodynamic partition function and it is shown that the thermodynamical properties of such a gas do not depend on the genus of the Riemann surface.

1. Introduction

Solitons are interesting objects to study and it is particularly interesting to study their dynamics. The moduli space approximation [6] gives an effective description of the dynamics of solitons at low energy when most of the degrees of freedom are frozen. The moduli space approximation works as follows: static multi-solitons are parametrized by the moduli space − the minima of the energy functional. At low energy, the actual field dynamics can be taken to be close to the moduli space, i.e. near the bottom of the valley of the energy functional. The dynamics projected onto the moduli space is then the geodesic motion on the moduli space [13]. For monopoles − solitons in three dimensions − the moduli space approximation has given important insight into the scattering and the bound states of the monopoles. It has found important application in proving various duality conjectures in supersymmetric field theories and in string theory [10].

On a plane, for Bogomol'nyi vortices in the Abelian Higgs model [1] − solitons in two dimensions − one can similarly describe their scattering [9]. For vortices on a compact Riemann surface, M, of genus g, one can do more − study their statistical mechanics [7]. Since the potential energy between the vortices is constant, in the moduli space approximation evaluation of the partition function of a gas of vortices effectively

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reduces to the computation of the volume of the moduli space. As the moduli space is Kähler, in order to find the volume, one needs to know the Kähler form or more precisely, its cohomology class. For the genus $g = 0, 1$ cases, the Kähler forms have been computed in [7, 12], respectively. For the sphere ($g = 0$) the N-vortex moduli space is the complex projective space CP_N. In this case symmetry arguments are enough to find the Kähler form. On the other hand, for the torus ($g = 1$), the Kähler form is found by exploiting the fibre bundle structure of the N-vortex moduli space. For genus $g \geq 1$ and $N \geq 2g - 1$, the N-vortex moduli space has a bundle structure, where the base is the Jacobian, J, of the Riemann surface, a torus of real dimension $2g$, and the fibre is CP_{N-g}. For $N \leq g$, the N-vortex moduli space is homeomorphic to a $2N$-dimensional analytic subvariety of the Jacobian. It would be interesting to find a general formula for the Kähler form and, hence, the volume of the moduli space for N vortices on an arbitrary Riemann surface, M, of genus g. Here, we will obtain such a formula.

In the next section we will see that the Kähler form is the sum of two parts: one is related to the Kähler form of M, the other is determined by the vortex interactions. The cohomology classes of both of these can be determined. This then enables the required formula for the volume of the moduli space to be computed. It depends on $N,\ g$ and the area of M. The various thermodynamical quantities for a gas of vortices can be deduced from this. It is found that the statistical mechanics of such a gas is independent of the genus of M. This is expected on physical grounds.

This paper is organized as follows. In Sect. 2, we briefly describe the Bogomol'nyi vortices and the Kähler form on the moduli space. In Sect. 3, we present the cohomological formula for the volume. Then we compare it with the previously computed cases for vortices on the sphere and the torus. This serves as a check of the volume formula. Finally, in Sect. 4, we compute the various thermodynamical quantities.

2. Vortices and the Kähler Form on the Moduli Space

(i) Bogomol'nyi vortices. Bogomol'nyi vortices are static, topologically stable, finite energy solutions of the critically coupled Abelian Higgs model in 2+1 dimensions [4]. We consider vortices on the space-time $\mathbf{R} \times M$, where M is a compact Riemann surface of genus g, and $\mathbf{R}$ parametrizes ordinary time x_0. The metric on $\mathbf{R} \times M$ can be taken to be of the form (locally)

$$ds^2 = dx_0^2 - \Omega(x_1, x_2)(dx_1^2 + dx_2^2), \tag{2.1}$$

where Ω is positive. The Lagrangian density of the model is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi\overline{D^\mu\phi} - \frac{1}{8}(|\phi|^2 - 1)^2, \tag{2.2}$$

where ϕ is a complex Higgs scalar field, A_μ is a $U(1)$ gauge potential, $D_\mu = \partial_\mu - iA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $(\mu, \nu = 0, 1, 2)$. The units are chosen such that both the gauge field coupling constant and the mass of the Higgs field are unity.

Working in the gauge $A_0 = 0$, the Lagrangian is $L = T - V$, where

$$T = \frac{1}{2}\int_M d^2x\,(\dot{A}_1\dot{A}_1 + \dot{A}_2\dot{A}_2 + \Omega\dot{\phi}\dot{\bar{\phi}}), \tag{2.3}$$

$$V = \frac{1}{2}\int_M d^2x\left(\Omega^{-1}F_{12}^2 + D_1\phi\overline{D_1\phi} + D_2\phi\overline{D_2\phi} + \frac{\Omega}{4}(|\phi|^2 - 1)^2\right) \tag{2.4}$$

are respectively the kinetic and the potential energies. Further, we need to impose Gauss's law. This arises from the equation of motion of A_0, as the following constraint,

$$\partial_1 \dot{A}_1 + \partial_2 \dot{A}_2 - \Omega\, Im(\dot{\phi}\overline{\phi}) = 0. \tag{2.5}$$

In the static case the total energy, $E = V$, can be reexpressed as [1]

$$E = \frac{1}{2}\int_M d^2x \left((D_1 + iD_2)\phi\,\overline{(D_1 + iD_2)\phi} + \Omega^{-1}\{F_{12} + \frac{\Omega}{2}(|\phi|^2 - 1)\}^2 + F_{12}\right). \tag{2.6}$$

Here we have omitted a total derivative term, which vanishes as M has no boundary. Bogomol'nyi vortices, which minimize the above energy integral, satisfy the first order Bogomol'nyi equations

$$(D_1 + iD_2)\phi = 0, \tag{2.7}$$

$$F_{12} + \frac{\Omega}{2}(|\phi|^2 - 1) = 0. \tag{2.8}$$

The solutions are classified into topologically stable sectors determined by the first Chern number [4, 14]

$$\frac{1}{2\pi}\int_M d^2x F_{12} = N, \tag{2.9}$$

where N is an integer. Note that, in general, there is an obstruction to the existence of N-vortex solutions on a compact surface. This is seen by integrating (2.8) over M. Since $\Omega|\phi|^2$ is non-negative, we deduce the bound, first obtained by Bradlow [2],

$$4\pi N \leq A, \tag{2.10}$$

where A is the area of M. Assuming that this is satisfied, the solutions with the first Chern number N are uniquely determined by specifying N zeros of the Higgs field [2, 14]. Thus, N can also be interpreted as the vortex number. Since vortices are indistinguishable, the vortex moduli space, M_N, is diffeomorphic to the symmetric product $(M)^N/S_N$, where S_N is the permutation group of N elements. It should be noted that M_N is a smooth manifold. In the sector with vortex number N, the potential energy of the vortices is $E = \pi N$.

It is possible to eliminate the gauge potentials from Eq. (2.8), by solving (2.7), thereby obtaining an equation for $|\phi|^2$,

$$\nabla^2 \log|\phi|^2 - \Omega(|\phi|^2 - 1) = 4\pi \sum_{i=1}^{N} \delta^2(\mathbf{x} - \mathbf{x}_i), \tag{2.11}$$

where $\mathbf{x}_i$ denotes the position of the zero of the Higgs field associated with the i^{th} vortex and $\nabla^2 = \partial_1^2 + \partial_2^2$.

The kinetic energy (2.3) induces a natural Riemannian metric on the moduli space [6]. Let q_α and $g_{\alpha\beta}(\mathbf{q})dq^\alpha dq^\beta$, where $(\alpha, \beta = 1, \cdots, 2N)$ denote real coordinates and the metric on M_N. Then, in the moduli space approximation for vortex motion the Lagrangian can be written as

$$L = \frac{\pi}{2} g_{\alpha\beta}(\mathbf{q})\dot{q}^{\alpha}\dot{q}^{\beta} - N\pi, \tag{2.12}$$

where π is the mass of a single vortex. Below, we shall use the analogue of this expression using complex coordinates for the vortex positions. Although we cannot determine $g_{\alpha\beta}$ explicitly, we shall show that it is possible to compute the total volume of M_N.

(ii) The Kähler form on the moduli space. Samols [9] has obtained an expression for the metric $g_{\alpha\beta}$ and the associated Kähler form on the N-vortex moduli space by analyzing data around the N zeros of the Higgs field assuming these are distinct. Detailed computation shows that the metric has a smooth extension to the complete moduli space, where vortices may coincide. Let z be a local complex coordinate on M. Let the vortex positions be $\{z_i = x_{1i} + ix_{2i} : i = 1, \cdots, N\}$. Since z_i is a simple zero of the Higgs field, log $|\phi|^2$ has the following series expansion obtained on using (2.11),

$$\begin{aligned} \log|\phi|^2 = \log|z-z_i|^2 + a_i + \frac{1}{2}b_i(z-z_i) + \frac{1}{2}\bar{b}_i(\bar{z}-\bar{z}_i) + c_i(z-z_i)^2 \\ - \frac{\Omega(z_i)}{4}(z-z_i)(\bar{z}-\bar{z}_i) + \bar{c}_i(\bar{z}-\bar{z}_i)^2 + \cdots . \end{aligned} \tag{2.13}$$

From the expression for the kinetic energy Eq. (2.3), Samols shows, after some integrations, that the metric is

$$ds^2 = \sum_{i,j=1}^{N} \left(\Omega(z_i)\delta_{ij} + 2\frac{\partial b_i}{\partial \bar{z}_j} \right) dz_i d\bar{z}_j. \tag{2.14}$$

Only the coefficients of the linear terms in (2.13) contribute to this formula. Notice that b_i is a function of the positions of all N vortices.

The reality property of the kinetic energy implies that

$$\frac{\partial \bar{b}_i}{\partial z_j} = \frac{\partial b_j}{\partial \bar{z}_i} \tag{2.15}$$

and from this follows the Hermiticity of the metric (2.14). One can then define the associated Kähler form as

$$\omega = \frac{i}{2} \sum_{i,j=1}^{N} \left(\Omega(z_i)\,\delta_{ij} + 2\frac{\partial b_i}{\partial \bar{z}_j} \right) dz_i \wedge d\bar{z}_j. \tag{2.16}$$

Using (2.15) one can show that ω is a closed (1,1) form. The volume of the moduli space is

$$\mathrm{Vol}_N = \frac{1}{N!} \int_{M_N} \omega^N. \tag{2.17}$$

The Kähler form ω can be divided into two parts $\omega = \omega_1 + \omega_2$, where

$$\omega_1 = \frac{i}{2} \sum_{i=1}^{N} \Omega(z_i) dz_i \wedge d\bar{z}_i \tag{2.18}$$

is just N copies of the area form induced from M and

$$\omega_2 = i \sum_{i,j=1}^{N} \frac{\partial b_i}{\partial \bar{z}_j} dz_i \wedge d\bar{z}_j \tag{2.19}$$

contains information about the relative vortex positions. Our aim is to understand the topological nature of ω_2 and its effect on Vol_N. If ω_2 is ignored, Vol_N would simply be $A^N/N!$. Notice that to obtain this result we have chosen a specific normalization of ω dictated by physics. In fact we can choose any normalization by multiplying the Lagrangian by an overall constant.

Notice that one can write $\omega_2 = -i\bar{\partial}B$, where B is a one-form of degree (1,0),

$$B = \sum_{i=1}^{N} b_i(z_1, z_2, \cdots, z_N, \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_N) dz_i. \tag{2.20}$$

Since z_i are natural coordinates on the Cartesian product $(M)^N$, not on the moduli space M_N, the symmetry of the one-form B is not manifest in the above equation. However, the indistinguishability of vortices implies that

$$b_i(\cdots, z_i, \cdots, z_j, \cdots) = b_j(\cdots, z_j, \cdots, z_i, \cdots)\ . \tag{2.21}$$

Thus, the one-form B is symmetric and hence, defined on M_N.

Before proceeding further we would like to point out that the one-form B has poles whenever $z_i = z_j$ for $i \neq j$. To see this let us consider the function ψ defined in a coordinate patch as follows:

$$\psi = \log|\phi|^2 - \sum_{i=1}^{N} \log|z - z_i|^2. \tag{2.22}$$

Notice that ψ is a smooth function, since the singularities of $\log|\phi|^2$ at the zeros of the Higgs field have been cancelled by the term $\sum_{i=1}^{N} \log|z - z_i|^2$. Then, as z_j approaches z_i, one can see that

$$b_i = \frac{2}{z_i - z_j} + \text{smooth part}, \tag{2.23}$$

hence, B has poles. It is useful to note that the residues of B are integers. This fact will be important later.

One simple way to uncover the topological significance of B is to determine its transformation properties under change of coordinates. Let us assume that M is covered in such a way that all N vortices lie in one coordinate patch U whose local coordinate is denoted by z. The i^{th} vortex in this patch has the coordinate z_i. Under a holomorphic coordinate transformation, U goes into another coordinate patch U'. In terms of the local coordinate $z \to z' = \zeta(z)$; and, also $z_i \to z'_i = \zeta(z_i)$. In the transformed coordinate, the expansion of $\log|\phi|^2$ in (2.13) reads

$$\begin{aligned}\log|\phi|^2 = \log|z' - z'_i|^2 + a'_i + \frac{1}{2} b'_i(z' - z'_i) + \frac{1}{2}\bar{b}'_i(\bar{z}' - \bar{z}'_i) + c'_i(z' - z'_i)^2 \\ - \frac{\Omega(z'_i)}{4}(z' - z'_i)(\bar{z}' - \bar{z}'_i) + \bar{c}'_i(\bar{z}' - \bar{z}'_i)^2 + \cdots .\end{aligned} \tag{2.24}$$

Here, $b'_i = b'_i(z'_1, \cdots, z'_i, \cdots, z'_N)$, writing out the coordinate dependence explicitly. Remember that $|\phi|^2$ is a globally well defined function on M. Thus, on the overlap region $U \cap U'$, by comparing the coefficients of $(z - z_i)$ on the right-hand sides of Eqs. (2.13) and (2.24) one finds

$$b_i = b'_i \frac{\partial \zeta_i}{\partial z_i} + \frac{\partial z_i}{\partial \zeta_i} \frac{\partial^2 \zeta_i}{\partial z_i^2}, \tag{2.25}$$

where $\zeta_i = \zeta(z_i)$. Notice the striking similarity with the corresponding transformation of the Levi–Civita connection of M

$$\Gamma^z_{zz} = \Gamma^{z'}_{z'z'} \frac{\partial z'}{\partial z} + \frac{\partial z}{\partial z'} \frac{\partial^2 z'}{\partial z^2}. \tag{2.26}$$

This heralds the topological nature of B. By looking at Eqs. (2.25) and (2.26), one concludes that B differs from the complex connection one-form on the co-tangent bundle of M_N by a globally defined one-form. Generically, this one-form is not smooth as it contains poles. If the poles were absent then $\omega_2 = -i\partial B$ would have been cohomologous to the complex Ricci curvature two-form of the Levi–Civita connection on the co-tangent bundle of M_N. This means that $\omega_2/2\pi$ would have been cohomologous to the first Chern class of the co-tangent bundle.

In what follows we will need to evaluate the integrals of ω_2 restricted to some special complex one-dimensional submanifolds. The integrals, as we will see shortly, receive two contributions: one is from the residues of B, and the other is due to the fact that B, restricted to these submanifolds, is related to the complex Levi–Civita connection.

First, we consider the submanifold of N coincident vortices. The solutions with N coincident vortices are parametrized by a complex one-dimensional submanifold, M_{co} of the moduli space M_N. M_{co} is diffeomorphic to M and lies inside the Jacobian J. Let Z be the position of the coincident vortices. $|\phi|^2$ now satisfies the equation

$$4\frac{\partial^2 \log|\phi|^2}{\partial z \partial \bar{z}} - \Omega(|\phi|^2 - 1) = 4\pi N \delta^2(z - Z) \tag{2.27}$$

and $\log|\phi|^2$ has the following series expansion around Z:

$$\log|\phi|^2 = N \log|z - Z|^2 + a + \frac{1}{2} b(z - Z) + \frac{1}{2}\bar{b}(\bar{z} - \bar{Z}) + \cdots \quad . \tag{2.28}$$

Then, the one-form B, restricted to M_{co}, simplifies to

$$B = b(Z, \bar{Z}) dZ. \tag{2.29}$$

By a similar analysis as in (2.25), one can determine the transformation properties of b under a holomorphic coordinate transformation $z \to \xi(z)$. One obtains

$$\frac{b}{N} = \frac{b'}{N} \frac{\partial \xi(Z)}{\partial Z} + \frac{\partial Z}{\partial \xi(Z)} \frac{\partial^2 \xi(Z)}{\partial Z^2}. \tag{2.30}$$

We remark that for N coincident vortices B does not contain any pole. By comparing (2.30) with (2.26), one finds that B/N restricted to the submanifold M_{co} differs from the complex Levi–Civita connection one-form of M_{co} by a smooth, globally defined one-form. Thus, ω_2/N restricted to M_{co} is cohomologous to the complex Ricci curvature two-form of the co-tangent bundle of M_{co}. Now, the volume of M_{co} is

$$V_{co} = \int_{M_{co}} \omega = \frac{iN}{2} \int_M \left(\Omega + 2\frac{\partial b}{\partial Z} \right) dZ \wedge d\bar{Z} = N\left(A - 4\pi N(1 - g)\right), \tag{2.31}$$

where use has been made of the Gauss–Bonnet formula for the integral of the curvature of the Levi–Civita connection, Γ^Z_{ZZ}, on M

$$\frac{-i}{2\pi}\int_M \left(\frac{\partial \Gamma^Z_{ZZ}}{\partial Z}\right) dZ \wedge d\bar{Z} = 2(1-g) \tag{2.32}$$

which implies

$$\frac{-i}{2\pi N}\int_M \left(\frac{\partial b}{\partial \bar{Z}}\right) dZ \wedge d\bar{Z} = 2(1-g). \tag{2.33}$$

Notice that $\frac{1}{2\pi}\int_{M_{co}} \omega_2 = 2N^2(g-1)$. The volume V_{co} agrees with the volumes previously computed for the sphere and the torus in [7, 12], respectively.

Secondly, let us consider two clusters of vortices with m and $(N-m)$ vortices, and let z_1 and z_2 be their positions on M, respectively. The solutions corresponding to such clusters are parametrized by a complex two-dimensional submanifold, M_c, of the moduli space M_N. We can do a similar analysis as in the above to compute the integral of ω_2 restricted to certain one-dimensional submanifolds of M_c. Restricted to M_c, B can be written as

$$B = b_1 dz_1 + b_2 dz_2. \tag{2.34}$$

Notice that b_1 has a pole at $z_1 = z_2$ and from the generalization of (2.23) to a pair of vortex clusters, one finds that $\mathrm{Res}(b_1) = 2(N-m)$. Following (2.25) one can determine the transformation properties of b_1 and b_2 under holomorphic coordinate changes $z_1 \to z_1'$ and $z_2 \to z_2'$. These are

$$b_1(z_1, z_2) = b_1'(z_1', z_2')\frac{\partial z_1'}{\partial z_1} + m\frac{\partial z_1}{\partial z_1'}\frac{\partial^2 z_1'}{\partial z_1^2}, \tag{2.35}$$

$$b_2(z_1, z_2) = b_2'(z_1', z_2')\frac{\partial z_2'}{\partial z_2} + (N-m)\frac{\partial z_2}{\partial z_2'}\frac{\partial^2 z_2'}{\partial z_2^2}. \tag{2.36}$$

We will be particularly interested in the case when the second cluster does not move, i.e. when z_2 is a constant. The vortex motion is then restricted to a one-dimensional submanifold, $\tilde{M}$, of M_c. $\tilde{M}$ is diffeomorphic to M.

Now, comparing (2.35) with (2.26) one sees that B/m, restricted to $\tilde{M}$, differs from the complex Levi–Civita connection one-form of $\tilde{M}$ by a one-form which contains a pole at $z_1 = z_2$. For the volume of $\tilde{M}$ one can write

$$\tilde{V} = \int_{\tilde{M}} \omega = I_r + I. \tag{2.37}$$

Here, I_r is the contribution coming from the residues and I contains the rest of the contribution. Similarly as in the derivation of (2.31) we find

$$I = m\,(A - 4\pi m(1-g)) \tag{2.38}$$

and the residue contribution is

$$I_r = -2\pi m\,\mathrm{Res}(b_1) = -4\pi m(N-m). \tag{2.39}$$

Thus, the total volume of $\tilde{M}$ is

$$\tilde{V} = m(A - 4\pi N + 4\pi mg). \tag{2.40}$$

As a consistency check, if $m = N$ then we have one cluster of N coincident vortices. In this case we get back (2.31) by simply putting $m = N$ in the above formula.

We remark that $\frac{1}{2\pi}\int_{\tilde{M}} \omega_2 = 2m(mg - N)$. Naturally, one would expect that the (1,1) form ω_2 belongs to $H^2(M_N, \mathbf{R})$, since this is a part of the Kähler form of M_N. However, because of the relationship between B and the complex Levi–Civita connection one-form, combined with the fact that the residues of B are integers, one sees that the integral of $\omega_2/2\pi$ over any complex one-dimensional submanifold is an integer. This means that $\omega_2/2\pi$ actually belongs to $H^2(M_N, \mathbf{Z})$. This information will be used in the next section in obtaining a cohomological formula for ω_2.

3. Cohomology and the Volume of the Moduli Space

(i) Cohomology ring of the symmetric products of a Riemann surface. Here, we quote several theorems without proof which will be used later. This also serves to fix the notation. The main reference is [5].

We have $H^0(M, \mathbf{Z}) = \mathbf{Z}$, $H^1(M, \mathbf{Z}) = \mathbf{Z}^{2g}$ and $H^2(M, \mathbf{Z}) = \mathbf{Z}$. Let α_i, $i = 1, \cdots, 2g$ be the generators of $H^1(M, \mathbf{Z})$ and β be the generator of $H^2(M, \mathbf{Z})$. It is useful to note that β is a normalized area form (i.e. its integral over M is unity) of type (1,1). The ring structure of $H^*(M, \mathbf{Z})$ can be described as follows:

$$\alpha_i\alpha_j = 0, \quad i \neq j \pm g \ ; \alpha_i\alpha_{i+g} = -\alpha_{i+g}\alpha_i = \beta, \quad 1 \leq i \leq g. \tag{3.1}$$

Here, juxtaposition means cup product. Let

$$\begin{aligned} \alpha_{ik} &= 1 \otimes \cdots \otimes 1 \otimes \alpha_i \otimes 1 \otimes \cdots \otimes 1 \in H^1((M)^N, \mathbf{Z}), \\ \beta_k &= 1 \otimes \cdots \otimes 1 \otimes \beta \otimes 1 \otimes \cdots \otimes 1 \in H^2((M)^N, \mathbf{Z}), \end{aligned} \tag{3.2}$$

the α_i and β being in the k^{th} place. Then, $H^*((M)^N, \mathbf{Z})$ is generated by the α_{ik} and the β_k $(1 \leq i \leq 2g, 1 \leq k \leq N)$ with the following relations being satisfied:

$$\begin{aligned} \alpha_{ik}\alpha_{jk} &= 0, \quad i \neq j \pm g, \\ \alpha_{ik}\alpha_{i+g,k} &= -\alpha_{i+g,k}\alpha_{ik} = \beta_k, \quad 1 \leq i \leq g, \\ \alpha_{ik}\alpha_{jl} &= -\alpha_{jl}\alpha_{ik}, \ k \neq l. \end{aligned} \tag{3.3}$$

Now, define the following symmetric linear combinations:

$$\begin{aligned} \xi_i &= \alpha_{i1} + \cdots + \alpha_{iN}, \quad 1 \leq i \leq 2g, \\ \eta &= \beta_1 + \cdots + \beta_N. \end{aligned} \tag{3.4}$$

Further, define $\xi_i' = \xi_{i+g}$ $(1 \leq i \leq g)$ and $\sigma_i = \xi_i\xi_i'$. Then we have the following result [5]

Theorem 1. *Let M be a compact connected Riemann surface of genus g, M_N its N^{th} symmetric product. Then, the cohomology ring $H^*(M_N, \mathbf{Z})$ is generated by elements $\xi_1, \cdots, \xi_g, \xi_1', \cdots, \xi_g'$ of degree 1, and an element η of degree 2, subject to the following relations:*

(a) the ξ's and ξ''s anti-commute with each other and commute with η;

(b) If $i_1, \cdots, i_a,\ j_1, \cdots, j_b,\ k_1, \cdots, k_c$ *are distinct integers from 1 to* g *inclusive, then*

$$\xi_{i_1} \cdots \xi_{i_a} \xi'_{j_1} \cdots \xi'_{j_b} (\sigma_{k_1} - \eta) \cdots (\sigma_{k_c} - \eta) \eta^q = 0 \tag{3.5}$$

provided that $a + b + 2c + q = N + 1$.

We will also need the following result on the cohomology of some particular submanifolds of M_N. Let $\nu = (N_1 \cdot p_1 + \cdots + N_k \cdot p_k)$ be a partition of N such that $p_1 > p_2 > \cdots p_k > 0$ and $N = \sum p_i N_i$. Then there exists a mapping from $\prod_{i=1}^k M_{N_i}$ onto a closed submanifold $\triangle(\nu)$ of M_N, where $\triangle(\nu)$ has N_1 clusters of p_1 coincident vortices, N_2 clusters of p_2 coincident vortices, etc. This mapping is an isomorphism. For any submanifold Y, let us write $[Y]$ for its cohomology class in $H^*(M_N, \mathbf{Z})$. Then, one can show that [5]

Theorem 2. $[\triangle(\nu)]$ *is the coefficient of* $\tau_1^{N_1} \cdots \tau_k^{N_k}$ *in*

$$P^{\rho - g} \eta^{N - \rho - g} \prod_{i=1}^{g} (P\eta + Q(\eta - \sigma_i)), \tag{3.6}$$

where

$$\begin{aligned} P &= p_1 \tau_1 + \cdots + p_k \tau_k, \\ Q &= (p_1^2 - p_1)\tau_1 + \cdots + (p_k^2 - p_k)\tau_k, \\ \rho &= N_1 + \cdots + N_k. \end{aligned} \tag{3.7}$$

Now, if $\delta_s = [\triangle(1 \cdot s + (N - s) \cdot 1)],\ s > 1$, so that δ_s is the cohomology class of the submanifold of M_N which consists of those points which have at least s vortices coinciding at one point, then one can show using Theorem 2 that

$$\delta_s = s(N + (g-1)(s-1))\eta^{s-1} - s(s-1)\eta^{s-2}(\sigma_1 + \cdots + \sigma_g). \tag{3.8}$$

In terms of the above notation, the submanifold M_{co} for N coincident vortices corresponds to $\triangle(1 \cdot N)$ and its cohomology class is

$$\delta_N = N(N + (g-1)(N-1))\eta^{N-1} - N(N-1)\eta^{N-2}(\sigma_1 + \cdots + \sigma_g). \tag{3.9}$$

Further, the total Chern class of the tangent bundle of M_N is $(1+\eta)^{N-2g+1} \prod_{i=1}^{g}(1 + \eta - \sigma_i)$. So, the first Chern class of the tangent bundle is

$$c_1(TM_N) = (N - g + 1)\eta - (\sigma_1 + \cdots + \sigma_g). \tag{3.10}$$

(ii) Cohomological formula for the Kähler form and the volume of the moduli space[1]. An expression for the cohomology class of the two-form ω_2 can be obtained using the fact that $\omega_2/2\pi$ is a (1,1) form belonging to $H^2(M_N, \mathbf{Z})$. Let us determine the generators of $H^2(M_N, \mathbf{Z})$ which are of type (1,1). One can see that η is a generator of $H^2(M_N, \mathbf{Z})$, and this is of type (1,1). The other type (1,1) generator of $H^2(M_N, \mathbf{Z})$ comes from the pairing of the generators of $H^1(M_N, \mathbf{Z})$. In Appendix (i) we show that it must be of the form

$$D'(\sigma_1 + \cdots + \sigma_g), \tag{3.11}$$

[1] An attempt to obtain a cohomological formula for the Kähler form was first made by P. Shah [11]. His work has inspired us to look further into the problem from a cohomological point of view.

where D' is a non-zero integer. Thus, the general expression for ω_2 reads

$$\omega_2 = 2\pi C(g,N)\eta + 2\pi D(g,N)(\sigma_1 + \cdots + \sigma_g), \tag{3.12}$$

where $C(g,N)$ and $D(g,N)$ are integers.

The coefficients $C(g,N)$ and $D(g,N)$ can be determined by computing the volumes of the submanifolds describing different types of coincident vortices by cohomological means and then comparing them with the same volumes previously computed in Sect. 2. The volume of M_{co} − which describes the motion of N coincident vortices − is

$$V_{co} = \int_{M_{co}} (\omega_1 + \omega_2) = \int_{M_N} (\omega_1 + \omega_2) \wedge \delta_N. \tag{3.13}$$

Using (3.9) and (3.5), one finds

$$V_{co} = N\left(A + 2\pi C(g,N) + 2\pi N g D(g,N)\right), \tag{3.14}$$

where we have used the fact that

$$\omega_1 = A\eta. \tag{3.15}$$

Equating this with (2.31), we require

$$C(g,N) + NgD(g,N) = 2N(g-1). \tag{3.16}$$

In Appendix (ii) we show that the volume of the submanifold $\tilde{M}$ − which describes the motion of m coincident vortices with the remaining $(N-m)$ vortices coincident and held fixed at a general position − is

$$\tilde{V} = m\left(A + 2\pi C(g,N) + 2\pi m g D(g,N)\right). \tag{3.17}$$

Comparing this with (2.40) gives

$$C(g,N) + mgD(g,N) = -2N + 2mg. \tag{3.18}$$

From (3.16) and (3.18), we find

$$C(g,N) = -2N, \quad D(g,N) = 2. \tag{3.19}$$

Thus, the Kähler form on M_N is

$$\omega = \omega_1 + \omega_2 = (A - 4\pi N)\eta + 4\pi(\sigma_1 + \cdots + \sigma_g). \tag{3.20}$$

Notice that

$$\omega_2/2\pi = -2c_1(TM_N) + 2(1-g)\eta, \tag{3.21}$$

where use has been made of (3.10). This shows that $\omega_2/2\pi$ is not just the first Chern class of the co-tangent bundle of M_N.

Now, putting all the ingredients together, and using (3.5), one finally obtains the following formula for the volume of the moduli space:

$$\mathrm{Vol}_N = \int_{M_N} \frac{\omega^N}{N!} = (A - 4\pi N)^{N-g} \sum_{i=0}^{g} \left(\frac{(4\pi)^i (A - 4\pi N)^{g-i} g!}{(N-i)!(g-i)!i!} \right). \tag{3.22}$$

In the formula above $N \geq g$. An analogous formula can be written for $N < g$. The sum now runs from $i = 0$ to $i = N$, and the factors of $A - 4\pi N$ are combined to give $(A - 4\pi N)^{N-i}$ in the sum. Notice that the volume is just a function of the area of M, its genus, and the number of vortices. It contains no information about the shape of M.

(iii) Examples: The volume of the moduli space for the sphere and the torus. For the sphere ($g = 0$), (3.22) gives

$$\mathrm{Vol}_N = \frac{(A - 4\pi N)^N}{N!}. \tag{3.23}$$

This is precisely the same as the formula obtained in [7]. On the other hand for the torus ($g = 1$) one gets

$$\mathrm{Vol}_N = \frac{A(A - 4\pi N)^{N-1}}{N!}. \tag{3.24}$$

Again this is the same as the formula obtained in [12]. Shah conjectured in [11] that the volume of the moduli space for any Riemann surface with genus $g > 1$ is given by (3.24). We, however, find this conjecture to be not true, e.g. for a Riemann surface of genus $g = 2$ and $N \geq 2$, the volume is

$$\mathrm{Vol}_N = \frac{(A^2 - 16\pi^2 N)(A - 4\pi N)^{N-2}}{N!} \tag{3.25}$$

which is different from (3.24).

4. Thermodynamics of the Vortices

Following [7], the thermodynamics of N vortices at temperature T can be treated using the Gibbs distribution. The partition function is

$$\mathcal{Z} = \frac{1}{h^{2N}} \int_{M_N} [d\mathbf{p}][d\mathbf{q}] e^{-E(\mathbf{p},\mathbf{q})/T}, \tag{4.1}$$

where h is Planck's constant, p_α are the momenta conjugate to the coordinates q_α and E is the energy. After doing the Gaussian momentum integrals, the partition function reduces to

$$\mathcal{Z} = (2\pi^2 T/h^2)^N \int_{M_N} [d\mathbf{q}](\det g_{\alpha\beta})^{1/2}. \tag{4.2}$$

The second factor in this partition function is just the volume, Vol_N, of the moduli space M_N.

Using (4.2) and (3.22) one obtains the partition function for a gas of N vortices on M

$$\mathcal{Z} = \frac{(A - 4\pi N)^{N-g}}{N!} \left(\frac{2\pi^2 T}{h^2}\right)^N R(g, A, N), \tag{4.3}$$

where

$$R(g, A, N) = \sum_{i=0}^{g} \frac{(A - 4\pi N)^{g-i}(4\pi)^i g! N!}{(N-i)!(g-i)!i!}. \tag{4.4}$$

To obtain the thermodynamic limit, we let $N \to \infty$, assuming that the density of the gas of vortices is a fixed constant given by $N/A = n$. Now, a short calculation shows that, at fixed n,

$$R(g, A, N) = A^g \left(1 + O(1/N)\right). \tag{4.5}$$

Using Stirling's formula for $N!$, when N is large, one obtains the free energy $F = -T \log \mathcal{Z}$,

$$F \simeq -NT \left(\log \frac{2e\pi^2 T}{h^2} - \log N + (1 - \frac{g}{N}) \log(A - 4\pi N) + \frac{g}{N} \log A + O(1/N) \right). \tag{4.6}$$

The pressure $P = -\partial F/\partial A$ is

$$P = \frac{NT}{A - 4\pi N}. \tag{4.7}$$

The entropy $S = -\partial F/\partial T$ is

$$S = N \left(\log \left(\frac{1 - 4\pi n}{n} \right) + \log \left(\frac{2e^2\pi^2 T}{h^2} \right) \right). \tag{4.8}$$

These are precisely the same formulae as obtained in [7, 12]. Notice that the genus g appears nowhere in the formulae for the thermodynamical quantities. Thus, the thermodynamics of a gas of vortices is independent of the topology of the space on which the vortices are moving.

5. Conclusion

Central to our study of the thermodynamics of a gas of vortices on an arbitrary Riemann surface is the computation of the volume of the vortex moduli space. The dependence of the volume on the area of the Riemann surface is quite noticeable. The area dependence disappears from the volume whenever $A = 4\pi N$ – Bradlow's limit. Then, for $N \leq g$ the volume of the moduli space is $\mathrm{Vol}_N = (4\pi)^N g!/[N!(g - N)!]$, and for $N > g$ the volume is zero. At $A = 4\pi N$ the Higgs field vanishes everywhere and the problem of solving the Bogomol'nyi equations reduces to the problem of solving for a constant magnetic field on the Riemann surface M. It can be shown that for $N = g$ the moduli space of this problem is related to the space of flat $U(1)$ connections on M. Time-varying flat connections have non-trivial kinetic energy, and hence, following the argument of Sect. 2, there is a metric on this moduli space. The volume of this moduli space is a topological quantity. It is of interest to see that the volume of this moduli space is equal to Vol_g at Bradlow's limit. This is shown in Ref. [8]. For $N > g$, it is also shown in [8], how Vol_N tends to zero as A approaches $4\pi N$.

Moduli spaces play an important role in diverse areas of physics and mathematics. In general it is desirable to know more about moduli spaces, e.g. their volume (compact cases), metric etc. Computation of the volume of a moduli space is not totally new. In [16], with a remarkable use of the Verlinde formula [15], Witten computed the volume of the moduli space of flat connections (for semi-simple gauge groups) on an arbitrary Riemann surface. In this case, however, the volume is a purely topological quantity. Thus, it is gratifying to see that in the case of the moduli space of Bogomol'nyi vortices on a compact Riemann surface one can also explicitly compute the volume. This is almost topological, but not exactly so, because the volume depends on the area of the Riemann surface, not on its shape.

Appendix

(i) A note on a (1,1) form belonging to $H^2(M_N, \mathbf{Z})$. Let $w_\rho,\ (\rho = 1, \cdots, g)$ be a basis of holomorphic one-forms on M with the period matrix $\Lambda = (\lambda_{\rho i}),\ (i = 1, \cdots, 2g)$. w_ρ is related to the generators α_i of $H^1(M, \mathbf{Z})$ as $w_\rho = \sum_{i=1}^{2g} \lambda_{\rho i}\alpha_i$. A basis of holomorphic one-forms on $(M)^N$ is given by

$$w_{\rho k} = 1 \otimes \cdots \otimes 1 \otimes w_\rho \otimes 1 \otimes \cdots \otimes 1,\ 1 \leq \rho \leq g,\ 1 \leq k \leq N \qquad \text{(A.1)}$$

with w_ρ being in the k^{th} place. Then, a basis ζ_ρ of holomorphic one-forms on M_N is given by the following symmetric linear combinations:

$$\zeta_\rho = w_{\rho 1} + \cdots + w_{\rho N},\ 1 \leq \rho \leq g. \qquad \text{(A.2)}$$

One sees that

$$\zeta_\rho = \sum_{i=1}^{2g} \lambda_{\rho i}\xi_i. \qquad \text{(A.3)}$$

Using the Riemann bilinear relations the period matrix can be written as $\Lambda^t = (I\ \ \Gamma)$, where I is the $(g \times g)$ unit matrix and $\Gamma = (\gamma_{jl}),\ (j, l = 1, \cdots, g)$ is a symmetric matrix with $Im(\Gamma) > 0$. Notice that under the diffeomorphisms of M the elements (γ_{jl}) can change.

Let $v \in H^2(M_N, \mathbf{Z})$ be expressed as

$$v = \frac{1}{2}\sum_{i,j=1}^{2g} q_{ij}\xi_i\xi_j, \qquad \text{(A.4)}$$

where $Q = (q_{ij})$ is an antisymmetric matrix with integer elements. Then, expressing v in terms of ζ_ρ one can show that it is of type (1,1) if the following constraint is satisfied [3]:

$$\Lambda^t Q^{-1}\Lambda = 0. \qquad \text{(A.5)}$$

This being a matrix constraint leaves one to freely choose g^2 elements of Q. However, for v to be invariant under diffeomorphisms of M, the above equation must be satisfied for arbitrary values of (γ_{jl}) with $Im(\gamma_{jl}) > 0$. This can be true only if Q has the following form:

$$Q = D' \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad \text{(A.6)}$$

where I is the $(g \times g)$ unit matrix and D' is a constant integer. Thus, on M_N any integral (1,1) form v must be of the following type:

$$v = D'(\sigma_1 + \cdots + \sigma_g). \qquad \text{(A.7)}$$

(ii) Proof of (3.17). Consider the mapping $j : M' \times M'' \to M_N$, given by $j(z_1, z_2) = (z_1, \cdots, z_1, z_2, \cdots, z_2)$, where z_1 occurs m times and z_2 occurs $(N - m)$ times. M' and M'' are two copies of M. For z_2 fixed, the mapping j is an isomorphism onto the submanifold $\tilde{M}$. One obtains

$$j^\star(\xi_i) = m\alpha_i' \otimes 1, \quad j^\star(\eta) = m\beta' \otimes 1. \qquad \text{(A.8)}$$

Here, α'_i and β' are, respectively, the generators of $H^1(M', \mathbf{Z})$ and $H^2(M', \mathbf{Z})$. Now,

$$\int_{\tilde{M}} \eta = \int_{M'} j^\star(\eta) = m \int_{M'} \beta' = m, \tag{A.9}$$

and, similarly, since $\sigma_i = \xi_i \xi_{i+g}$,

$$\int_{\tilde{M}} \sigma_i = m^2, \ 1 \leq i \leq g. \tag{A.10}$$

Thus,

$$\int_{\tilde{M}} \omega = m(A + 2\pi C(g, N) + 2\pi m g D(g, N)) \tag{A.11}$$

as claimed.

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